Mathematical Reviews

Vol. 18, No. 11 DECEMBER, 1957 pp. 863-982

FOUNDATIONS, THEORY OF SETS, LOGIC

Ehrenfeucht, A.; and Mostowski, A. Models of axiomatic theories admitting automorphisms. Fund. Math. 43 (1956), 50–68.

Let S be an axiomatic theory, M a model of S with X as domain of individuals and let G_M be the group of automorphisms of M. A group G_1 of transformations of a set X_1 strongly contains a group G of transformations of a set X if $X_1 \supset X$ and each $f \in G$ can be extended to at least one function $f_1 \in G_1$. It is shown that each theory S which has at least one infinite model has a model M_0 such that G_{M_0} strongly contains an infinite cyclic group. More generally, if a theory S has at least one infinite model, then for each ordered set X there is a model M_0 of S such that G_{M_0} strongly contains the group of transformations of X onto itself which leave the ordering invariant. The proofs use the following theorem due to Ramsey [Proc. London Math. Soc. (2) 30 (1929), 264–286]: Let Y be an infinite set and Y^n the set of subsets of Y having exactly n elements. If $Y^n = C_1 \cup \cdots \cup C_k$ is a partition of Y^n into mutually disjoint sets, then there is a $j \leqq k$ and an infinite set $Y_1 \subseteq Y$ such that $Y_1^n \subseteq C_j$. *L. N. Gál.*

Schwabhäuser, Wolfram. Über die Vollständigkeit der elementaren euklidischen Geometrie. Z. Math. Logik Grundlagen Math. 2 (1956), 137–165.

This paper is a complement to Tarski's "A decision method for elementary algebra and geometry" [2nd ed., Univ. of California, 1951; MR 13, 423]. Let ARZ be the axiomatic system of elementary algebra, as given by Tarski, and DEG that of three-dimensional euclidean geometry, consisting of Hilbert's axiom groups I–IV, completed by a Dedekind axiom. Let $\mathfrak{R}$ be the field of real numbers (or any algebraically closed field) and $\mathfrak{G}$ the three-dimensional analytic geometry based on $\mathfrak{R}$. Tarski obtained a decision method for DEG by correlating to every formula H of DEG a formula rdz H of ARZ, so that (i) H is true in $\mathfrak{G}$ if and only if rdz H is true in $\mathfrak{R}$. In order to prove the completeness of DEG, the author correlates to every formula K of ARZ a formula rdg K of DEG; roughly speaking, rdg K is the result of interpreting the arithmetical operations by those of Hilbert's "Streckenrechnung". He now proves (ii) K is provable in ARZ if and only if rdg K is provable in DEG; (iii) in DEG, H is equivalent to rdg rdz H. Thus, if H is true in $\mathfrak{G}$, then by (i) and Tarski's completeness result for ARZ, rdz H is provable in ARZ, by (ii) rdg rdz H is provable in DEG, and by (iii) the same holds for H. *A. Heyting.*

Darkow, M. D. Interpretations of the Peano postulates. Amer. Math. Monthly 64 (1957), 270–271.

Harrop, R. On disjunctions and existential statements in intuitionistic systems of logic. Math. Ann. 132 (1956), 347–361.

It is well-known that in the intuitionistic predicate calculus a formula $A \vee B$ is provable only if either A or B is, and a formula without free variables is provable only if some $A(a)$ is. This paper gives a new proof of this for the intuitionistic propositional algebra, and extends it to a formalism of elementary arithmetic, based on intuitionistic predicate calculus, like that given in Kleene's "Introduction to metamathematics" [Van Nostrand, New York, 1952; MR 14, 525] § 19. A modification for the intuitionistic predicate calculus is sketched; it does not give, as Schütte [Math. Ann. 122 (1950), 47–65; MR 12, 233] does, a recursive method of constructing the simpler proof. The gist of the author's method is as follows. He first replaces modus ponens by a series of rules, giving directly the result of applying modus ponens one or more times to each axiom scheme, such that the conclusion has a definite structure; thus, with the axiom scheme

$$(X \supset Y) \supset ((X \supset (Y \supset Z)) \supset (X \supset Z))$$

he associates the rules

$$\frac{X \to Y}{(X \supset (Y \supset Z)) \supset (X \supset Z)}, \quad \frac{X \supset Y,\ X \supset (Y \supset Z)}{X \supset Z},$$

but not any rule leading to a simple Z. The theorems of the resulting system are then well ordered in a series T^* in order of increasing complexity; and there is then a deletion process which deletes from T^* any compound formula which is not semantically justified by its predecessors (e.g. $X \supset Y$ is deleted if and only if X is present but not Y; $(a)X(a)$ if one of the instances $X(n)$ is missing, etc.). He then shows that there are actually no deletions. There are, of course, complexities when quantifiers are present, and the process is non-constructive. *H. B. Curry* (University Park, Pa.).

Orey, Steven. Formal development of ordinal number theory. J. Symb. Logic 20, 95–104 (1955).

Ordinal number theory as developed in Quine's New Foundations NF results in some strange theorems [e.g. Th. XII. 3.15 in Rosser's "Logic for mathematicians", McGraw-Hill, New York, 1953; the abbreviations found in this book are used here; MR 14, 935]. In the larger system of mathematical logic ML stronger versions of the relevant definitions of similarity and well-ordering suggest themselves, hence it is natural to ask whether these lead to theorems which agree with ordinal number-theory as usually developed. Roughly speaking these new definitions in ML arise by allowing universal quantifiers to range over all sets and classes, while in NF they can only range over sets. For example, if variables denoted by small letters range over sets, those denoted by capital letters over both sets and classes, then we call X strongly well-ordered (abbreviated WORD(X)) if

$$\mathrm{Sord}(X).(Y): Y \cap \mathrm{AV}(X) \neq \Lambda . \supset .(Ez).z = \min_X (Y),$$

while adopting the definition in NF we would have had: X is weakly well-ordered (Word(X)) if

$$\mathrm{Sord}(X).(y): y \cap \mathrm{AV}(X) \neq \Lambda . \supset .(Ez).z = \min_X y,$$

as new class of strongly well-ordered ordinals ORN we now take $\hat{x}((Ey).\mathrm{WORD}(y).x=\mathrm{No}(y))$, instead of NO, defined as $\hat{x}((Ey).\mathrm{Word}(y).x=\mathrm{No}(y))$. The new class ORN is a subclass of NO and is more satisfactory in some respects. The Burali-Forti paradox is avoided, for although ORN$\subseteq$NO and NO ϵ **V**, we have ORN $\not\epsilon$ **V**. However it turns out that if ML is consistent, then one cannot prove in ML that ω_0 ϵ ORN. Let ML′ be the system which results from ML by adding 'ω_0 ϵ ORN' as an axiom. Even then there still seems to be no way of proving in ML′ that ω_1 ϵ ORN. Hence we strengthen the system to ML″ by adding the axiom A: $(\alpha):\alpha\ \epsilon\ \mathrm{ORN}\ .\supset.\ \omega_\alpha\ \epsilon\ \mathrm{ORN}$. In ML″ all of conventional ordinal arithmetic can be developed. It is shown that in place of A one could for example, also have added the following axiom E:

$$(x)(Y):x\subseteq \mathrm{ORN}.Y\subseteq \mathrm{ORN}.x\mathrm{SM}\ Y.\supset.(E\alpha).\alpha\ \epsilon\ \mathrm{ORN}.Y\subseteq\hat{\beta}(\beta<\alpha).$$

By adding a new predicate Orn to NF so that 'Orn(A)' is an (unstratified) statement for every term A, and some new axioms, one sees that there are also extensions of NF in which an ordinal theory analogous to the theory of ORN can be developed. *L. N. Gál* (Ithaca, N.Y.).

Orey, Steven. On the relative consistency of set theory. J. Symb. Logic **21** (1956), 280–290.

[The notation used is that of the paper reviewed above and J. B. Rosser, Logic for mathematicians, McGraw-Hill, New York, 1953; MR **14**, 935.] Let ML″ be the system of ML together with axiom E defined in the above review, and Σ the system of set-theory by Gödel [The consistency the continuum hypothesis, Princeton, 1940; MR **2**, 66]. By a modification of the construction described there one can build a model of Σ in ML″, thus proving the consistency of Σ relative to ML″. Variations of this method show that if NF* is the system NF together with the ι-operator and axioms for the ι-operator, and also axioms

$$(1)\quad (x)(y):x\subseteq \mathrm{NO}.y\subseteq \mathrm{NO}.x\ \mathrm{sm}\ y.(E\alpha)\alpha\ \epsilon\ \mathrm{NO}.x\subseteq\hat{\beta}(\beta\subset\alpha).\supset.(E\gamma).\gamma\ \epsilon\ \mathrm{NO}.y\subseteq\hat{\beta}(\beta<\gamma),$$

$$(2)\quad (\beta):\beta\ \epsilon\ \mathrm{NO}.\supset.(E\alpha).\beta<\omega_\alpha.\omega_\alpha\ \epsilon\ \mathrm{NO},$$

then NF* contains a model for Σ, and that there is another extension NF″ of NF in which we can obtain a model of Zermelo set-theory Σ'. The latter construction can be formalized in ML″, so the statement expressing the consistency of Σ' can be proved in ML″. *L. N. Gál.*

Mendelson, Elliott. The independence of a weak axiom of choice. J. Symb. Logic. **21** (1956), 350–366 (1957).

Let H be the following axiom of choice: For every denumerable set x of disjoint 2-element sets there is a set y which has exactly one element in common with each element of x; let D_j ($j=2, 3, \cdots$) be the statement $\sim(\exists x_1, \cdots, x_j)(x_1\epsilon x_2\epsilon\cdots\epsilon x_j\epsilon x_1)$, and let G be a set theory with Gödel's axioms A, B, C, and also all the D_j's. The main result of this paper states that H is not a theorem of G. This result was also obtained quite independently by J. R. Shoenfield [J. Symb. Logic **20** (1955), 202]. Both Mendelson's and Shoenfield's proofs are based on ideas of Fraenkel and Mostowski for a system with Urelements [cf., e.g., Mostowski, Fund. Math. **31** (1939), 201–252], and in both the basic new idea is the use of infinite descending ϵ-sequences in place of Urelements. In carrying out the proof, Mendelson first shows that if G is consistent, then so is the system G' with axioms A, B, C, E, D_j ($j=2, 3, \cdots$) and Q, where Q is an axiom which states that there exists a sequence of disjoint sequences such that each member of any of the sequences is the unit class of the next member of the sequence. The proof proceeds by constructing a model of G' starting from a model of $G\frown\{E\}$. Next, given a model of G', let $\Delta_0, \Delta_1, \cdots$ be the disjoint infinite descending sequences whose existence is asserted by Q, let $a=\{\Delta_0, \Delta_1, \cdots\}$, $b=\mathfrak{S}(a)$, V_c = closure of b under the power-set operation. It is possible to define a subclass $\mathfrak{F}\subseteq V_c$ which will be a model of G and in which H does not hold. Shoenfield's (unpublished) proof consists in the construction of a model of $G\frown\{\sim H\}$ directly from a model of G. Q holds in this model, too, so the consistency of G' also follows. Here, however, the universe V_s of the model is built up from b by transfinite induction in a manner similar to the construction of the class L by Gödel. Both authors actually obtain the stronger result: The well-ordering theorem is independent of the total-ordering principle (every set can be linearly ordered). *L. N. Gál* (Ithaca, N.Y.).

Mendelson, Elliott. Some proofs of independence in axiomatic set theory. J. Symb. Logic **21** (1956), 291–303.

Using a method of proof due to Firestone and Rosser [same J. **14** (1949), 79], the author here obtains new proofs of the following theorems: (a) The existence of inaccessible ordinals is not provable from the axioms of set-theory, if these axioms are consistent. (b) The axiom of infinity is independent of the other axioms, if these are consistent. (c) The axiom of replacement is independent of the other axioms, if these are consistent. In all cases, $V=L$ is included as an axiom. Theorem (a) was first proved by Kuratowski [Ann. Soc. Polon. Math. **3** (1924), 146–147], and theorem (b) and (c) without the addition of '$V=L$' by Bernays [J. Symb. Logic **13** (1948), 65–79; MR **10**, 3]. To sketch the proofs, let ϕ be the statement we wish to show unprovable from the other axioms, F Gödel's generating function of the class L of constructible sets, $L_\beta=F''\beta$, where β is an ordinal. For each ϕ corresponding to one of the theorems (a), (b), (c) it is possible to choose a suitable β such that L_β satisfies all the axioms of set-theory except ϕ, and moreover such that if ϕ were provable from the remaining axioms, then the consistency of L_β could be proved in L_β. This would lead to a contradiction, so ϕ cannot be provable from the remaining axioms, if they are consistent.

L. N. Gál (Ithaca, N.Y.).

Świerczkowski, S. On cyclic ordering relations. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 585–586.

A three-argument relation $[x, y, z]$ on the integers is called a cyclic ordering relation if (I) for distinct x, y, z, exactly one of $[x, y, z]$, $[x, z, y]$ holds; (II) if $[x, y, z]$, then $[z, x, y]$; (III) if $[x, y, z]$ and $[z, u, x]$, then $[u, y, z]$; and IV) if $[x, y, z]$, then $[x+k, y+k, z+k]$ for every integer k. A partial cyclic ordering relation $[x, y, z]_N$ is defined as above on a finite set $N=\{0, 1, \cdots, n\}$. Theorem 1. Every partial cyclic ordering relation can be extended to a cyclic ordering relation. Theorem 2. For every relation $[x, y, z]_N$, there are at most three values of x_2-x_1, where $x_1, x_2\ \epsilon\ N$ and are such that $[x_1, y, x_2]_N$ fails for all $y\ \epsilon\ N$. Like theorems involving the reals mod 1 are also announced. The proofs of all these results will appear in Annales Polonici Mathematici.

L. Gillman.

Mihailescu, Eugen. Formes normales dans l'ensemble $S(C)$. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 8 (1956), 329–361. (Romanian. Russian and French summaries)

[For notations see Mihailescu, Acad. R. P. Romîne. Stud. Cerc. Mat. 2 (1951), 1–44; MR 16, 554.] The author considers forms of different types built with the help of the operator C, and shows that they allow forms built with the help of the operators E and A. There are in total six theorems concerning this. We consider here only the first of them, the normal forms involved in the other theorems being too lengthy to be reproduced here. The author first deduces $Cpq \sim EqAqp$ and with the help of this formula proves that any form of the type

$$C^{n-1}p_1p_2\cdots p_n,$$

where C^n means $\underbrace{CC\cdots C}_{n}$ (similar notations are used for the other operators), allows the normal form

$N_1(C)=$

$$E^{n-1}p_n(Ap_np_{n-1})(A^2p_np_{n-1}p_{n-2})\cdots(A^{n-1}p_np_{n-1}\cdots p_2p_1).$$

B. Germansky (Jerusalem).

Smith, E. C., Jr.; and Tarski, Alfred. Higher degrees of distributivity and completeness in Boolean algebras. Trans. Amer. Math. Soc. 84 (1957), 230–257.

If A is a Boolean algebra, then A is called α-complete, α a cardinal number, if the sum of every subset of A of power at most equal to α exists in A. A is called complete if it is α-complete for every cardinal number α. α-completeness and completeness of ideals are defined analogously. A is called (α, β)-distributive, α and β cardinal numbers, if for any set of elements $\{a_{\xi,\eta}\}$, $a_{\xi,\eta}\in A$, where ξ and η range over index sets of power α and β respectively, written $\xi<\alpha$, $\eta<\beta$, we have

$$\prod_{\xi<\alpha}\sum_{\eta<\beta} a_{\xi,\eta}=\sum_{f\in\beta^\alpha}\prod_{\xi<\alpha} a_{\xi,f(\xi)}$$

and where β^α is the set of all functions f assigning to each value $\xi<\alpha$ a value $f(\xi)<\beta$, provided that all sums and products exist. A is called completely distributive if it is (α, β)-distributive for every α and β. If X is a subset of A, then $\kappa(X)$ denotes the power of X and $\delta(X)$ stands for the least cardinal ρ, such that the power of every disjointed subset of X (disjointed: $xy=0$ for any x and $y\in X$, $x\neq y$) is at most ρ. First, the authors prove several theorems concerning (α, β)-distributivity. If α, α', β and β' are cardinal numbers, $2\leqq\alpha'\leqq\alpha$, $2\leqq\beta'\leqq\beta$, then (α, β)-distributivity implies (α', β')-distributivity. If $\alpha'\leqq\alpha$, $\beta'\leqq\beta$, then the question is considered under which conditions (α', β')-distributivity implies (α, β)-distributivity. A well-known result is that every Boolean algebra is $(2, \beta)$-distributive for every cardinal number β, but the authors are able to prove that $(\alpha, 2)$-distributivity implies (α, α)-distributivity. Again it is shown that (α, α)-distributivity implies $(\alpha, 2^{(\alpha)})$-distributivity, provided that A is $2^{(\alpha)}$-distributive, where $2^{(\alpha)}$ means the αth cardinal arithmetical power of 2. A corollary is that every (α, β)-distributive Boolean algebra is (α, α)-distributive. A cardinal β is called a limit number if there is no largest cardinal among all those smaller than β. β is a strong limit number if $2^{(\alpha)}<\beta$ for every $\alpha<\beta$. β is a singular cardinal if it is the sum of fewer than β cardinals, each of which is smaller than β. A regular cardinal is a cardinal which is not singular. It is proved that if β is a strong limit cardinal and A is β-complete and (α, β)-distributive for every $\alpha<\beta$, then A is (β, β)-distributive. An example is given of a β-complete Boolean algebra, β a singular strong limit cardinal, which is (α, α)-distributive for every $\alpha<\beta$ and which, for some $\alpha<\beta$, is not (α, β)-distributive. A Boolean algebra is called atomistic if the unit element is the sum of the atoms. The equivalence of several statements concerning atomisticity of a Boolean algebra is proved. Among these, we mention the following: 1) A is atomistic, 2) A is completely distributive, 3) A is (β, β)-distributive for some β and $\delta(A)<\beta$, 4) A is isomorphic to a field of sets which is β-complete in the wider sense (i.e. the sum of β sets is the set-theoretic union). The following section of the paper is primarily devoted to a discussion of sufficient conditions for ideals in Boolean algebras to be α-complete. Let, for every cardinal α, α^* denote the least upper bound of the set of cardinals $2^{(\xi)}$, $\xi<\gamma$. It is known that a proper principal ideal in A can be extended to an α-complete prime ideal in A if and only if A is isomorphic to an α-complete field of sets. The following theorem is proved. If A is a (γ, γ^*)-distributive α-complete Boolean algebra, $\alpha\geqq\gamma^*$ and I is an α-complete proper ideal in A, such that $\delta(A-I)<\gamma$, then there exists an α-complete prime ideal P in A, such that $I\subseteq P$. Moreover if A is β-complete and I is not β-complete, then there exists such a P which also is not β-complete. If I is an ideal in A, then it is evident that $\delta(A-I)\leqq\delta(A/I)$. It is proved that the equality sign holds if and only if I is $\delta(A-I)$-complete. A cardinal number β is called strongly attainable from a cardinal number α, if there is no regular limit number ξ such that $\alpha<\xi\leqq\beta$, and β is called weakly attainable from α if there is no regular strong limit number ξ, such that $\alpha<\xi\leqq\beta$. It is proved that if A is a complete atomistic Boolean algebra, $\kappa(A)=\beta$, I an ideal in A, such that $\sum I=1$, $\delta(A-I)\leqq\alpha$, β strongly attainable from α, then there exists a set $M\subseteq I$, $\kappa(M)\leqq\alpha$ and $\sum M=1$. Again, if A is complete and atomistic, $\kappa(A)=\beta$, $\sum I=1$, $\delta(A-I)<\gamma$ for some cardinal γ such that $\alpha\geqq\gamma^*$ and β is weakly attainable from α, then there exists a set $M\subseteq I$, $\kappa(M)\leqq\alpha$ and $\sum M=1$. If A is a β-complete Boolean algebra, then it is not difficult to show, that if X is a subset of A, $\kappa(X)\leqq\beta$, then there exists a disjointed subset Y of A, $\kappa(Y)\leqq\beta$, such that $\sum X=\sum Y$. The following stronger result is proved. If I is an ideal in a Boolean algebra A, such that the sum of every disjointed subset Y of I of at most β elements exists in A, then the sum of every subset of I of at most β elements exists in A. The reviewer comments on the proof, that this also can be given without using the distributive law, which means that the theorem also holds for any relatively complemented lattice. Indeed, using the notations of the authors, let $y_0=x_0$ and $y_\xi=\sum_{\eta\leqq\xi}x_\eta-\sum_{\eta<\xi}x_\eta$, where the right hand member denotes the relative complement of $\sum_{\eta<\xi}x_\eta$ in $[0, \sum_{\eta\leqq\xi}x_\eta]$, then one can prove that for every ρ, $0<\rho<\alpha$, one has $\sum_{\eta\leqq\xi}x_\eta=\sum_{\eta\leqq\xi}y_\eta$ and then again that $\sum_{\eta<\alpha}x_\eta$ exists and is equal to $\sum_{\eta<\alpha}y_\eta$. Using the above result the authors derive sufficient conditions in order that an ideal in a β-complete Boolean algebra be β-complete. We only mention the following results. If A is β-complete, I α-complete, β weakly attainable from α, $2^{\delta(A-I)}\leqq\alpha$, then I is β-complete. If A is β-complete, I α-complete, β strongly attainable from α, then I is β-complete. It also follows that if A is β-complete, I α-complete and prime, β weakly attainable from α, then I is β-complete. Some other theorems closely related to this subject are proved. The last part of the paper is devoted to questions concerning completeness of factor algebras. It follows immediately from the definition of β-complete-

ness of ideals, that if A and I are β-complete, that A/I is β-complete. If moreover $\delta(A/I)\leqq\beta$, then it follows from a previous theorem, that every subset of A/I has a sum and thus A/I is complete. The authors also prove the following stronger result. If A is β-complete and I α-complete for every $\alpha<\beta$ and $\delta(A/I)\leqq\beta$, then A/I is complete. In particular, it follows that if A is σ-complete and $\delta(A/I)\leqq\aleph_0$, then A/I is complete. This result yields a measure theoretic application. Let m be a finite finitely additive measure on a Boolean algebra A and

$$I=\{x|m(x)=0,\ x\in A\},$$

then it is known that $\delta(A/I)\leqq\aleph_0$. Then it follows from the preceding results that A/I is complete. Finally some known results on the existence of countably additive measures on A/I and in particular on the existence of countably additive two-valued measures on A/I, if A is the Boolean algebra of all subsets of a set S, are extended to more general Boolean algebras. If I is a countably complete proper ideal in a complete Boolean algebra A and $\delta(A)$ is strongly attainable from $\aleph_0$ and $\delta([a])>\aleph_0$ for every $a\neq 0$, $a\in A$, then there is no countably additive measure on A/I. If A is complete, I countably complete and proper, $\delta(A)$ weakly attainable from $\aleph_0$, $\sum I=1$, then there is no countably additive two-valued measure on A/I.

Ph. Dwinger (W. Lafayette Ind.).

Rieger, Ladislav. On certain fundamental problems of mathematical logic. Časopis Pěst. Mat. **81** (1956), 342–351. (Czech)

Mihăilescu, Eugen. Formes normales dans le calcul des propositions bivalentes. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **8** (1956), 297–327. (Romanian. Russian and French summaries)

[Concerning notations see Mihăilescu, Acad. R. P. Romîne. Stud. Cerc. Mat. **2** (1951), 1–44; MR **16**, 554.] The author begins with the axioms

(1) $CCCpqCrsCtCCspCrp$,

(2) $CEpqCpq,\ CEpqCqp,\ CCpqCCqpEpq$,

(3) $CKpqp,\ CKpqq,\ CpCqKpq$,

(4) $EERpqRrsEEpqErs$,

(5) $EKRpqrRKprKqr$.

He proves that the subsystems of propositional algebra $\mathbf{L}(C, E)$ consisting of the consequences of the axioms (1) and (2), $\mathbf{L}(C, E, K)$ consisting of the consequences of the axioms (1), (2) and (3), and $\mathbf{L}(C, E, K, R)$ consisting of the consequences of the axioms (1)–(5) are complete. The method of proof is, in the cases of $\mathbf{L}(C, E, K)$ and $\mathbf{L}(C, E, K, R)$ that of normal forms.

B. Germansky (Jerusalem).

Schröter, Karl. Über den Zusammenhang der in den Implikationsaxiomen vollständigen Axiomensysteme des zweiwertigen mit denen des intuitionistischen Aussagenkalküls. Z. Math. Logik Grundlagen Math. **2** (1956), 173–176.

Proof of the following theorem: Let M_0 be a system of axioms for the implication-theorems of the intuitionistic propositional calculus I and M_0^* for those of the classical calculus K; let $M_0\cup M_1$ be an axiom system for I (or a part of I); then $M_0^*\cup M_1$ is an axiom system for K (or the corresponding part of K). This yields the theorem originally proved by the author and stated without proof by Hermes and Scholz [Enzykl. math. Wiss., Bd. II, H. 1, T. I, Teubner, Leipzig 1952, p. 37, (f); MR **16**, 435].

A. Heyting (Amsterdam).

Goodstein, R. L. The decision problem. Math. Gaz. **41** (1957), 29–38.

An expository lecture.

Asser, Günter. Über die Ausdrucksfähigkeit des Prädikatenkalküls der ersten Stufe mit Funktionalen. Z. Math. Logik Grundlagen Math. **2** (1956), 250–264.

It is shown that for any formula of the lower predicate calculus with identity and with functional terms (IFK) a formula can be found of the calculus without functional terms (IK) which will be satisfiable in a model if and only if the first formula is satisfiable. From this theorem and well-known theorems for IK (completeness theorem, for example) corresponding theorems for IFK are proven. Some similar results are obtained for the lower predicate calculus with functional terms but without identity (PFK) by providing for any formula H of PFK a relation which can play the role of identity for the predicates of H.

P. C. Gilmore (University Park, Pa.).

Rose, Alan. A Gödel theorem for an infinite-valued erweiterter Aussagenkalkül. Z. Math. Logic Grundlagen Math. **1**, 89–90 (1955).

The infinite-valued predicate calculus considered here has as truth-values all real numbers x, $0\leqq x\leqq 1$, with 1 as the designated truth-value. The primitives are implication C, a special ternary function G and the universal quantifier $\prod$. $Gpqr$ is defined by giving a rule for determining its truth-value $g(x, y, z)$ when p, q, r have the values x, y, z resp. When $p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_n$ take the truth-values $x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n$, resp., then the truth-value of $\prod p_i\, \Phi(p_1, \cdots, p_n)$ is the greatest lower bound of the truth-values of $\Phi(p_1, \cdots, p_n)$ as x_i varies $(i=1, \cdots, n)$. It is shown that there is no plausible complete formalization of this calculus. *L. N. Gál* (Ithaca, N.Y.).

Mučnik, A. A. On the separability of recursively enumerable sets. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 29–32. (Russian)

Two sets E_1, E_2 of natural numbers are called strongly inseparable; if they fulfill the following conditions: 1) $E_1\cap E_2=\emptyset$; 2) for any recursively enumerable set M, if M/E_1 is infinite, then $M\cap E_2\neq\emptyset$; if $M\cap E_2$ is infinite, then $M/E_1\neq\emptyset$; 3) $N\backslash(E_1\cup E_2)$ is infinite. The author constructs two strongly inseparable sets. As strongly inseparable sets cannot be effectively inseparable, this solves Uspenski's problem of constructing recursively inseparable, but not effectively inseparable sets. As they cannot be universal recursively enumerable sets, the result also solves Novikov's problem of constructing non-universal recursively inseparable sets. Some further results are deduced, of which I mention the following: If E is one of two strongly inseparable sets, then E is simple in no recursively enumerable set. (The definition of a set simple in another set is obvious.) A subset E of N^k is called homomorphic to a subset H of N^1, if there exists a mapping φ of N^k onto N^1, so that $\varphi(E)=H$, $\varphi(N^k\backslash E)=N^1\backslash H$. Then a recursively enumerable set, which is homomorphic to a simple set, is recursively inseparable from every recursively enumerable set which it does not overlap. Several unsolved problems are stated.

A. Heyting (Amsterdam).

Friedberg, Richard M. Two recursively enumerable sets of incomparable degrees of unsolvability (solution of Post's problem, 1944). Proc. Nat. Acad. Sci. U.S.A. 43 (1957), 236–238.

Construction of two arithmetical functions, both of which represent recursively enumerable sets, and neither of which is recursive in the other. *A. Heyting.*

See also: Monteiro, p. 867; Diener, p. 868; Lombardi, p. 879.

ALGEBRA

Combinatorial Analysis

Shia, Wen-Hou. A general formula for circular permutations. Amer. Math. Monthly 64 (1957), 347–348.

In this note, we establish a general formula for the number of circular permutations of a set of n objects containing similar elements. *From the introduction.*

Hall, Marshall, Jr. An algorithm for distinct representatives. Amer. Math. Monthly 63 (1956), 716–717.

An algorithm is suggested, suitable for programming on a digital computer, for constructing a system of distinct representatives for a finite system of (possibly overlapping) finite sets, or for verifying that no such system of distinct representatives exists. *P. Hall.*

Crampin, Joan. On note 2449. Math. Gaz. 41 (1957), 55–56.

Some constructions for flexagons. *W. T. Tutte.*

See also: Slater and Lakin, p. 888.

Elementary Algebra

Lemaître, G. Le calcul élémentaire. Acad. Roy. Belg. Bull. Cl. Sci. (5) 42 (1956), 1140–1145.

L'utilisation de la „partition binaire" permet d'effectuer les opérations élémentaires de l'arithmétique, sans nécessiter la connaissance des tables d'addition et de multiplication. *Résumé de l'auteur.*

Linear Algebra

Mendel, C. W.; and Barnett, I. A. A functional independence theorem for square matrices. Pacific J. Math. 6 (1956), 709–720.

A proof of the following theorem is given: Let A be an n-square matrix whose elements are independent indeterminates over an arbitrary field; then the Jacobian of the n traces of $A, A^2, \cdots, A^n$ with respect to each set of n distinct elements of A, at least one of which is a diagonal element, is never identically zero in the n^2 elements of A. Matrices are exhibited for which the Jacobians do not vanish. *B. N. Moyls* (Vancouver, B.C.).

See also: Freudenthal, p. 871; Marcus and Lopes, p. 877; Rinehart, p. 882; Čermák, p. 905; Lyusternik, p. 937; Mackenzie, p. 937; Pötzl, p. 970; Pipes, p. 971; Hoffman and Kruskal, p. 980.

Polynomials

Miller, Richard A. A Pascal triangle for the coefficients of a polynomial. Amer. Math. Monthly 64 (1957), 268–269.

Butler, M. C. R. Reducibility criteria for polynomials of two general classes. Proc. London Math. Soc. (3) 7 (1957), 63–74.

Let R be a field, $a, b, c \in R$, let $F_m(x) \equiv x^{2m} - ax^m + b$, and let $\phi_m(x, y)$ be the unique polynomial for which $x^m + y^m \equiv \phi_m(x+y, xy)$. The author obtains complete criteria for the reducibility over R of the polynomials $F_m(x)$ and $\phi_m(x, c) - a$. In the special case when $b = c^m$ there exists a Tschirnhaus transformation T under which each root of $\phi_m(x, c) - a$ is the image of exactly two roots of $F_m(x)$. Under the assumption that $F_m(x)$ is separable, the author determines conditions under which T is reversible. *J. A. Todd* (Cambridge, England).

Petersson, Hans. Über eine Zerlegung des Kreisteilungspolynoms von Primzahlordnung. Math. Nachr. 14 (1955), 361–375 (1956).

Let q be a prime >3 and let $f(u)$ be the polynomial whose roots are the primitive qth roots of unity. The author studies the decomposition of $f(u)$ in the quadratic field Ω determined by $\sqrt{\pm q}$, where $\pm q \equiv 1 \pmod 4$. In this field he factors $f(u) = f_1(u) f_2(u)$ and expands each factor $f_i(u)$ in powers of $u-1$: $f_i(u) = \sum_{m=0}^{r} \lambda_m^{(i)}(u-1)^m$, $r = \frac{1}{2}(q-1)$, $\lambda_r^{(i)} = 1$, $i = 1, 2$. He gives a detailed procedure for finding the coefficients $\lambda_m^{(i)}$ and relates some of them to other arithmetic functions of q. For example, if $q \equiv 1 \pmod 4$, then $\lambda_0^{(1)} = g^{\frac{1}{2}} \varepsilon^{-h}$ ($q^{\frac{1}{2}} > 0$), where ε is the fundamental unit and h the class number of Ω. The well-known facts that $N(\varepsilon) = -1$ and h is odd are an almost immediate consequence. The author derives the relation just mentioned, as well as a similar one for the case where $q \equiv 3 \pmod 4$, by a simple application of the Dirichlet formula for h. Then the ratios $\lambda_m^{(i)}/\lambda_0^{(i)}$ are related to invariantso f the quadratic form $\frac{1}{2}(x_1^2 + x_2^2 \cdots + x_k^2)$ for certain odd values of k. In this way he is able to obtain relations between h and the number of representations of q as the sum of 5 or 7 squares.

H. W. Brinkmann (Swarthmore, Pa.).

See also: Hammersley, p. 941.

Partial Order Structures

Monteiro, Antonio. Axiomes indépendants pour les algèbres de Brouwer. Rev. Un. Mat. Argentina 17 (1955), 149–160 (1956).

The author defines a Brouwer algebra as a lattice with 0 and an additional binary operation $\rightarrow$ satisfying the following two postulates: B1. If $a \wedge x \leq b$, then $x \leq a \rightarrow b$. B2. $a \wedge (a \rightarrow b) \leq b$. The system is called a generalized Brouwer algebra if it lacks 0, and an implicative system if it also lacks the operation $\vee$ (and so is a semilattice). The author gives an elementary and detailed proof that the following six axioms form an independent set characterizing a Brouwer algebra as an algebraic system with intuitive equality with the three binary operations $\rightarrow$, $\wedge$, $\vee$, and the constant element 0: A1. $a \rightarrow a = b \rightarrow b$; A2. $(a \rightarrow b) \wedge b = b$; A3. $a \wedge (a \rightarrow b) = a \wedge b$; A4. $a \rightarrow (b \wedge c) =$

$(a\to b)\wedge(a\to c)$; A5. $(a\vee b)\to c=(a\to c)\wedge(b\to c)$; A0. $0\wedge a=a$. Further the axioms A1–A5 characterize a generalized Brouwer algebra, and the axioms A1–A4 characterize an implicative system. *H. B. Curry* (University Park, Pa.).

Schmidt, Jürgen. Zur Kennzeichnung der Dedekind-MacNeilleschen Hülle einer geordneten Hülle. Arch. Math. **7** (1956), 241–249.

An isotone mapping ϕ from a partly ordered set E into a complete lattice F will be called "σ-normal", when the inverse image of every normal (alias closed) ideal in F is normal in E; it will be called "σ-true" if $y\geqq\phi(M)$ implies $y\geqq\phi(\sup M)$, whenever $\sup M$ exists in E; σ-normal mappings are σ-true. A subset E of a complete lattice F will be called σ-true or σ-normal, if the identity mapping of E into F is σ-true resp. σ-normal; it will be called "σ-dense", if every $x\in F$ satisfies $x=\sup M$ for some subset M of E. Dually, one can define the terms δ-normal, δ-true, and δ-dense. The main conclusion is that, for F to be the Dedekind-MacNeille completion of E, each of the following pairs of conditions is necessary and sufficient: (i) E is σ-dense and δ-dense; (ii) E is σ-dense and δ-normal; (iii) E is δ-dense and σ-normal. *G. Birkhoff.*

Diener, Karl-Heinz. Über zwei Birkhoff-Frinksche Struktursätze der allgemeinen Algebra. Arch. Math. **7** (1956), 339–345.

Büchi [Portugal. Math. **11** (1952), 151–167; MR **14**, 940] has defined an element a of a complete lattice L to be untranscendable, if $a\leqq\bigvee g$ ($g\in G$, a directed set) implies $a\leqq g_0$ for some $g_0\in G$. Let $a\leqq b$ in a complete lattice L, and let c be any untranscendable element of L contained in b but not in a. The author shows that the set M of all elements between a and b which fail to contain c, contains a maximal element m_0. He then applies this result to fill in a gap in the proof, by the reviewer and Frink, of a characterization of the class of lattices of all subalgebras of abstract algebras [Trans. Amer. Math. Soc. **64** (1948), 299–316, Th. 2; MR **10**, 279]. *G. Birkhoff.*

Ohkuma, Tadashi. Sur quelques ensembles ordonnés linéairement. Fund. Math. **43** (1956), 326–337.

A simply ordered set or chain may be called simply homogeneous, if the group of its order-automorphisms is simply transitive. It is shown that any simple homogeneous chain is isomorphic with an additive subgroup of the real number system. The number of isomorphism-types of simply homogeneous chains is shown to be 2^c, where c is the power of the continuum. *G. Birkhoff.*

Peremans, W. Embedding of a distributive lattice into a Boolean algebra. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 73–81.

Every distributive lattice D can be embedded in a Boolean algebra B, because it is isomorphic to a ring of sets. The paper describes in detail an algebraic construction of such an embedding which does not appeal to the axiom of choice. An attempt at such a construction was made by MacNeille [Bull. Amer. Math. Soc. **45** (1939), 452–455], but the present author points to a gap in the proof which he has not been able to fill. His own construction proceeds on different lines. To begin with he adjoins to D a new 0 and 1, obtaining a distributive lattice D'. Next he forms the set V of all pairs (a, b) with a and $b\in D'$ and then the set W of non-empty finite subsets of V. (It is helpful to imagine the embedding completed and to interpret (a, b) as $a\frown b'$, where the dash denotes the complement.) The elements of W are split into equivalence classes under finite chains of three types of elementary identifications. These equivalence classes are candidates for election to membership of a Boolean algebra $B(D)$. The bulk of the paper is concerned with suitable definitions of lattice operations among them and with the technical details of proving that $B(D)$ is, in fact, a Boolean algebra with the right sort of complements and with an isomorphic image of D as a sublattice. Any Boolean algebra containing D as a sublattice also contains a homomorphic image of $B(D)$; the homomorphism can deviate from an isomorphism only in connection with the greatest and least elements of D, if they exist, and the ambiguity is very slight. Finally the author shows a connection (or rather lack of connection) between his results and those of Dilworth [Trans. Amer. Math. Soc. **57** (1945), 123–154; MR **7**, 1]. Dilworth has proved that every lattice can be embedded in a lattice in which every element has a unique complement. But when his construction is applied to a distributive lattice with more than one element, one never obtains a distributive lattice.

K. A. Hirsch (London).

See also: Smith and Tarski, p. 865; Whitesitt, p, 869; Johnson, p. 869; Bandyopadhyay, p. 871; Zacher, p. 871; Halperin and Luxemburg, p. 909.

Rings, Fields, Algebras

★ **Snapper, Ernst. Integral closure of modules and complete linear systems. Algebraic geometry and topology.** A symposium in honor of S. Lefschetz, pp. 167–176. Princeton University Press, Princeton, N. J., 1957. $7.50.

A purely algebraic proof is given of an analogue of the fact that any linear system of divisors on an algebraic variety is contained in a complete linear system. The result is: Let Σ be a finitely generated extension field of the field k, L a finite k-submodule of Σ, and define the integral closure $|L|_i$ of L to consist of all $\sigma\in\Sigma$ for which there exists a non-zero finite k-submodule M_σ of Σ such that $\sigma M_\sigma\subset M_\sigma L$ (this is equivalent to the existence of an equation $\sigma^m+a_1\sigma^{m-1}+\cdots+a_m=0$, where $a_j\in L^j$ for $j=1, \cdots, m$). Then $|L|_i$ is a finite k-module.

M. Rosenlicht (Evanston, Ill.).

Terpstra, Fedde J. On a set of rings contained in a field of rational functions. Math. Ann. **133** (1957), 41–51.

Let $K=k(x, y)$ be a pure transcendental extension of an algebraically closed field k, and let Σ be the set of rings $P=\{u, v\}$ of elements $A(u, v)/B(u, v)$, where u and v are generators of K/k, A and B are polynomials, and $B(0, 0)\neq 0$. When Σ is ordered by inclusion each element P has immediate successors. The principal theorem in this paper asserts that all immediate successors of P are obtained by quadratic transformations centered at the maximal ideal (u, v) in P. Proofs are based on properties of rings of formal power series.

This material is part of the theory of birational correspondences between algebraic surfaces which has been extensively developed by O. Zariski. The main results obtained in the present paper are known. It appears that the author was not aware of this since he makes no reference to the literature. *H. T. Muhly.*

Nagata, Masayoshi. A remark on the unique factorization theorem. J. Math. Soc. Japan 9 (1957), 143–145.

It is known that if K is a field of characteristic different from 2, and if $n \geqq 5$, then $K[x_1, \cdots, x_n]/(\sum_{i=1}^{n} x_i^2)$ is a unique factorization ring [cf., e.g., van der Waerden's Einführung in die algebraische Geometrie, Springer, Berlin, 1939]. The author proves a generalization of this theorem which yields a simpler proof of the original.

M. Henriksen (Lafayette, Ind.).

Rees, D. Filtrations as limits of valuations. J. London Math. Soc. 32 (1957), 97–102.

Let A be a Noether domain; a homogeneous filtration on A is defined to be function f on the non-zero elements of A such that (i) $f(x)$ is a non-negative real number, (ii) $f(x-y) \geqq \min[f(x), f(y)]$, (iii) $f(xy) \geqq f(x)+f(y)$, and (iv) $f(x^n)=nf(x)$. The author shows that for any homogeneous filtration f on A there exists a sequence of discrete valuations v_n on the field of fractions F of A such that for $a \neq 0$ in A, (1) $v_n(a)$ is a non-negative integral multiple of $m(n)^{-1}$, where $m(n)$ is an integer depending on n, and (2) $f(a)=\liminf v_n(a)$. If f is a rational-valued valuation on F which is non-negative on A, then (2) can be strengthened to: (2′) $f(a)=\lim v_n(a)$ for all $a \neq 0$ in F.

When F is a finitely generated field extension of transcendence degree d of a field k, and f is a rational valuation of F over k, a suitable domain A can be found in F; in this case the residue class field of each approximant v_n has transcendence degree $d-1$ over k.

B. N. Moyls (Vancouver, B.C.).

Whitesitt, J. Eldon. Construction of the lattice of complemented ideals within the unit group. Pacific J. Math. 6 (1956), 779–794.

The rings considered in this paper are about of same type as the rings of endomorphisms of linear manifolds (F, A) with char. $(F) \neq 2$ [cf. R. Baer, Linear algebra and projective geometry, Academic Press, New York, 1952; MR 14, 675]. It is shown that important properties of those rings are determined entirely within the multiplicative group of the units of the ring. Notation: R=ring with an identity element 1; if $R=A \oplus A'$ and A, A' are right ideals, they are "complemented right ideals" (C.I.R.). U, Z, C, N denote unit subgroup, centre, centraliser and normaliser respectively, e is an idempotent $\neq 0$, u, v, w involutions, $s=uv$ is an "element of class 2", if $u \neq v$, $(S-1)^2=0$. R is supposed to satisfy: 1) $r \to r+r$ is an automorphism of the additive group R. 2) The set of the C.I.R.'s is closed for $\frown$ and $\smile$. 3) If $k \in R$, either eRk, or kRe implies $k=0$. 4) $Z(U(eRe)) \leqq Z(eRe)$. — The elements of class 2 are characterised by the following theorem (stated by the author to be due to I. Halperin): $s=uv$ $(u \neq v, u, v \neq \pm 1)$ is of class 2 if and only if for some $r \in U$ and for some involution w, $wsw=s^{-1}$, $rsr^{-1}=s^2$, $C(w) \leqq C(r)$, $C^2(s) \frown C(w) \leqq Z(U)$ and either $s^3 \neq 1$, or $(s')^3=1$ for every $s'=u'v'$. If A is a C.I.R., the set of all the involutions u such that $ux=x$ (resp. v such that $vx=-x$) for every $x \in A$ is denoted by $\Delta^+(A)$ (resp. $\Delta^-(A)$). Δ means Δ^+ and Δ^-. There is a reciprocal correspondence between the systems (Δ^+, Δ^-) in $U(R)$ and the C.I.R.'s. These form an irreducible complemented modular lattice which is completely determined by the order relation among the structures $\Delta(A)$ and $N\Delta(A)$ within U. The necessary and sufficient conditions for X lying between A and B (both $\neq 0$ and R) are $\Delta(A) \leqq N\Delta(B)$, or equivalently $\Delta(B) \leqq N\Delta(A)$ and $1 < [\Delta(A)^2 \frown \Delta(B)^2] \leqq \Delta(x)^2$.

F. W. Levi (Berlin).

Johnson, R. E. Structure theory of faithful rings. I. Closure operations on lattices. II. Restricted rings. Trans. Amer. Math. Soc. 84 (1957), 508–522, 523–544.

This begins a series of papers concerned with the structure of faithful rings (i.e., rings $\mathfrak{R}$ with no nonzero left annihilator $\mathfrak{R}^l$) satisfying various minimum conditions.

Part I develops the lattice-theoretic background; it is limited to complete lattices $\mathfrak{L}$ in which

$$(\sum_i A_i) \frown A = \sum_i (A_i \frown A)$$

for every chain $\{A_i\} \subset \mathfrak{L}$ and $A \in \mathfrak{L}$ (e.g., the lattice of all right ideals of any ring). Sample results: Subset $\mathfrak{A}$ of $\mathfrak{L}$ is called an inset if $I \in \mathfrak{A}$ and $\mathfrak{A}$ is closed under $\frown$; if a is a closure operation on $\mathfrak{L}$ then the set $\mathfrak{L}^a$, of all $A \in \mathfrak{L}$ such that $A^a=A$, is an inset of $\mathfrak{L}$. The lattice $\mathfrak{C}(\mathfrak{L})$ of closure operations on $\mathfrak{L}$ is dually isomorphic to the lattice of insets of $\mathfrak{L}$, under the correspondence $a \to \mathfrak{L}^a$. If a closure operation a is a $\frown$-endomorphism of $\mathfrak{L}$ then a is called an m-closure operation on $\mathfrak{L}$; the set of all such is a complete sublattice of $\mathfrak{C}(\mathfrak{L})$. Closure operations involving atoms of $\mathfrak{L}^a$ are studied. A closure operation a on a modular lattice with $O^a=O$ is called reducible if for every $A \in \mathfrak{L}^a$, $A \neq I$, there exists a nonzero $B \in \mathfrak{L}$ with $A \frown B=O$. An m-closure a with $O^a=O$ is reducible if and only if $\mathfrak{L}^a$ is a complemented modular lattice; then a is unique. If x is a $\smile$-homomorphism of lattice $\mathfrak{L}$ into lattice $\mathfrak{M}$ with $xO=O$, then x^{-1} (taking the maximal inverse image) is a $\frown$-homomorphism of $\mathfrak{M}$ into $\mathfrak{L}$.

Part II applies the results of I to the global structure theory of a faithful ring and its modules — in particular to direct sum representations, closure operations on modules, annihilators, and "restricted rings". The latter name is applied to $\mathfrak{F}''$ if it is complete, where $\mathfrak{F}''$ is the set of all right ideals $\mathfrak{A}$ with $\mathfrak{A} \frown \mathfrak{A}^l=O$ and $\mathfrak{A}=\mathfrak{A}^{ll}$; $\mathfrak{F}''$ is a Boolean algebra. In studying such rings, an important role is played by "large" ideals, that is, ideals $\mathfrak{S}$ with $\mathfrak{S}^l=O$. Many of the results, especially on direct sums, generalize to faithful rings facts already found by the author for semi-prime rings [same Trans. 76 (1954), 375–388; MR 16, 5]. *P. M. Whitman* (Silver Spring, Md.).

★ **van Leeuwen, Leonardus. Ring extension theory.** Dissertation, Technical Institute, Delft, 1957. 82 pp.

An extension theory for rings is given in a manner analogous to the extension theory for groups given by R. Baer [Math. Z. 38 (1934), 375–416]. The highly technical nature of the development prevents the giving of details in a review of reasonable size. *M. Henriksen.*

Drazin, M. P. A generalization of polynomial identities in rings. Proc. Amer. Math. Soc. 8 (1957), 352–361.

Let $\pi(\xi)$ (i.e., $\pi(\xi_1, \cdots, \xi_t)$) be a monomial in some indeterminates $\xi_1, \cdots, \xi_t$. Let P_π be all monomials which are not of the form $\pi\pi_1$; and let S_π be all those in P_π which have degree not less than that of π. Suppose for every $r_1, \cdots, r_t$ in a ring R it happens that $\pi(r)$ lies in the right ideal generated by the $\rho(r)$ for $\rho \in P_\pi$ [or merely for $\rho \in S_\pi$]. Then the author calls π a pivotal [or strongly pivotal, respectively] monomial for R. These always exist when R is an algebraic algebra of bounded degree, has a polynomial identity, or satisfies either minimal condition. After

developing the concept somewhat, the author shows (among other things) that, R having a pivotal monomial, R modulo its Jacobson radical is a subdirect sum of complete matrix rings of bounded order over division rings. *R. Arens* (Los Angeles, Calif.).

San Soucie, R. L. Weakly standard rings. Amer. J. Math. 79 (1957), 80–86.

A ring R is called weakly standard provided it is flexible and satisfies the identities: $((w, x), y, z)=0$, $(w, (x, y), z)=0$. Suppose that R is a prime ring or a simple ring or a primitive ring; then R is weakly standard if and only if R is either associative or commutative. Suppose that R is a simple flexible ring; then commutators are in the left nucleus of R if and only if R is either associative or commutative. Some of these results overlap the concurrent work of Kleinfeld [Canad. J. Math. 8 (1956), 335–340; MR 17, 1180]. Several pertinent examples are cited.

R. A. Good (College Park, Md.).

Cartier, Pierre. Une nouvelle opération sur les formes différentielles. C. R. Acad. Sci. Paris 244 (1957), 426–428.

The paper gives first a certain homomorphism φ_1 of the exterior algebra $\Omega^*(K)$ over the K-space $\Omega^1(K)$ of k-differentials of a commutative algebra K over a field k of characteristic p into the homology ring of the complex $(\Omega^*(K), d)$, where d is an anti-derivation extending $d: K\to\Omega^1(K)$; φ_1 associates x, dx $(x\in K)$ with the classes of x^p, $x^{p-1}dx$. In case K is a field extension (or more generally in case $k\subset K^p$ and K has a p-basis) φ_1 is an isomorphism, and for $\omega\in\Omega^*(K)$, with $d\omega=0$, $C(\omega)$ is defined to be the image by φ_1^{-1} of the class of ω. Besides some elementary formulae we have

$$\langle C(\omega), D\rangle^p=\langle\omega, D^p\rangle-D^{p-1}\langle\omega, D\rangle,$$

where D is a k-derivation of K. $C(\omega)=\omega$ gives a necessary and sufficient condition for $\omega\in\Omega^*(K)$ with $d\omega=0$ to have a form dx/x $(x\in K)$; for the proof an analogy of E. Noether's lemma in Galois theory is used. These results, which generalize some results of Jacobson [Trans. Amer. Math. Soc. 42 (1937), 206–224] and Tate [Proc. Amer. Math. Soc. 3 (1952), 400–406; MR 13, 905], are applied to a complete normal curve X over an algebraically closed field k: For every differential form ω over X and for every point x of X we have $\operatorname{res}_x C(\omega)=(\operatorname{res}_x\omega)^p$, and the k-space $\Omega^1(k(X))$ is the direct sum of $\bigcup_{m\geq 0}\partial(W_m(k(X))$ and the space generated by df/f $(f\in k(X))$, where $W_m(k(X))$ is the ring of Witt vectors $(x_0, \cdots, x_{m-1})$ over $k(X)$ and ∂ is defined by

$$\partial(x_0, \cdots, x_{m-1})=\sum_{i=0}^{m-1}x_i^{p^{m-i-1}-1}dx.$$

It follows that the matrix A of Hasse and Witt [Monatsh. Math. Phys. 43 (1936), 477–492], which is defined either by the p-behavior of primitive functions of g non-special points, g being the genus, or by the expansions of differentials of 1st kind at g non-special points and which has significance for both the group of divisor classes and unramified cyclic extensions of $k(X)$, is that of the map $D\to D^p$ in the Lie algebra of the Jacobian variety of X. Applications are made also to complete normal varieties and abelian varieties. *T. Nakayama* (Nagoya).

See also: Diener, p. 868; de Groot, p. 871; Carlitz, p. 875; Gel'fand, p. 913; Gonshor, p. 915; Vagner, p. 926; Kastler, p. 973.

Groups, Generalized Groups

McLain, D. H. Remarks on the upper central series of a group. Proc. Glasgow Math. Assoc. 3 (1956), 38–44.

Let $1=Z_0\leq Z_1\leq\cdots\leq Z_\alpha\leq\cdots$ be the upper central series of a group G and $1=F_0\leq F_1\leq\cdots\leq F_\alpha\leq\cdots$ the upper FC-series of G (so that $F_{\alpha+1}/F_\alpha$ is the set of all those elements of G/F_α that have only a finite number of conjugates). The terminal members at which the two series become stationary are the hypercentre H and the hyper-FC-subgroup F. Clearly $F_\alpha\geq Z_\alpha$ for every α. The main result of the paper is the following theorem. If the centre of G is torsion-free, then $F_\alpha\cap H=Z_\alpha$ for every α; in particular, in a ZA-group (uppernilpotent group) with torsion-free centre $F_\alpha=Z_\alpha$ for every α. If G is only assumed to be locally nilpotent, then the author shows that it is still true that $H=F$ and hence obtains the corollary: In a locally nilpotent group with torsion-free centre $F_\alpha=Z_\alpha$ for every α.

When finiteness conditions are imposed, further relations arise. The following three conditions on F turn out to be equivalent: Finitely generated, maximal condition for subgroups, maximal condition for normal subgroups. The corresponding result for H is due to Baer [Math. Z. 59 (1953), 299–338; MR 15, 598]. In addition F is then a finite extension of a finitely generated nilpotent group. The author gives an example to show that the minimal condition for the normal subgroups of F (or even H) need not imply the minimal condition for all subgroups. But he proves that if G itself satisfies the minimal condition for normal subgroups, then H and F satisfy the minimal condition for subgroups.

Finally, the author constructs examples of groups that are swept out by their ascending central series of precise length α, where α is an arbitrarily preassigned ordinal number. Examples of a different nature have been given by Gluškov [Mat. Sb. N.S. 31(73) (1952), 491–496; MR 14, 617]. *K. A. Hirsch* (London).

Gerstenhaber, Murray. On canonical constructions. II. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 881–883.

In part I [same Proc. 41 (1955), 233–236; MR 17, 581] the author described a general method of proving that all automorphisms of a mathematical structure are inner automorphisms. In the present paper the type of structure studied is the group S' of all homeomorphisms of a topological space S onto itself. For the case where S is an arc, the theorem that all automorphisms of the group of homeomorphisms are inner was proved by N. J. Fine and G. E. Schweigert [Ann. of Math. (2) 62 (1955) 237–253; MR 17, 288]. The author gives a new proof of this theorem in terms of his general theory of canonical constructions.

O. Frink (University Park, Pa.).

Haimo, Franklin. Semi-direct products with ample homomorphisms. Trans. Amer. Math. Soc. 84 (1957), 401–425.

"Semi-direct product"="complemented extension"= group G with a factorisation HK, where H is normal and H and K have trivial intersection. Such a semi-direct product determines, and is determined by, a homomorphism ϕ of K into the automorphism group of H, and ϕ is said to be ample if its image include the inner automorphisms of H. The paper under review gives a large number of theorems on semi-direct products, some of which require special conditions, the most frequent being that ϕ is ample, and that H is nilpotent of class 2. The sorts of problem that

the author is interested in are: the construction of automorphisms of G; the construction of the standard characteristic subgroups of G, especially its upper central series (of which a precise, but complicated, description is given if ϕ is ample); and the determination of the centraliser of H, and the normaliser tower of K in G. Special attention is given, as an example, to "inclusion products"; these are defined, if either H is a normal subgroup of K, or K is a subgroup of H, by the following multiplication in the Cartesian product of H and K:

$$(h_1, k_1)(h_2, k_2) = (h_1k_1h_2k_1^{-1}, k_1k_2).$$

Graham Higman (Oxford).

Benado, Mihail. Über die allgemeine Theorie der regulären Produkte von Herrn O. N. Golowin. I. Math. Nachr. **14** (1955), 213–234 (1956).

A group G is a regular product of a set of its subgroups G_i if G is the group-theoretic union of the G_i and if each G_i has only the trivial intersection with the normal subgroup generated by the rest of the subgroups of the set. Each such decomposition of G corresponds to a set of orthogonal, idempotent endomorphisms, the group-theoretic union of the image spaces of which is G, a Fitting endomorphism system [see Golovin, Mat. Sb. N.S. **27(69)** (1950), 427–454; MR **12**, 672]. A regular product of groups G_i, where each G_i is a regular product of groups H_{ij}, is a regular product of all the H_{ij}. The author proves that a necessary and sufficient condition for a regular product to be completely associative is that the kernel of the mapping which carries the corresponding free product onto the given regular product is admissible under all members of all Fitting endomorphism systems of all factorizations of the free product. R. R. Struik has independently found examples of regular products which are non-associative [Trans. Amer. Math. Soc. **81** (1956), 425–452; MR **17**, 1051]. The author also shows that the smallest normal subgroup in a free product of groups G_i which contains all commutators from pairs of the G_i's is a free group, generalizing the result of Golovin [loc. cit., II, 1.14 Theorem]. *F. Haimo* (St. Louis, Mo.).

de Groot, J. Equivalent abelian groups. Canad. J. Math. **9** (1957), 291–297.

Abelian groups $\mathfrak{G}$ and $\mathfrak{H}$ are equivalent if $\mathfrak{G}$ is isomorphic to a subgroup $\mathfrak{G}'$ of $\mathfrak{H}$, and $\mathfrak{H}$ is isomorphic to a subgroup $\mathfrak{H}'$ of $\mathfrak{G}$. It is known that equivalent groups $\mathfrak{G}$ and $\mathfrak{H}$ need not be isomorphic even if $\mathfrak{G}'$ and $\mathfrak{H}'$ are direct summands of $\mathfrak{H}$ and $\mathfrak{G}$ respectively. In this article the author presents a condition (each ascending sequence of direct summands of $\mathfrak{G}$ is a direct summand of $\mathfrak{G}$) which with the equivalence of $\mathfrak{G}$ and $\mathfrak{H}$ implies an isomorphism of $\mathfrak{G}$ and $\mathfrak{H}$. That equivalent complete (or divisible) groups (also equivalent additive groups of division rings) are isomorphic is obtained as a corollary. When $\mathfrak{G}$ possesses a base it is shown that equivalence implies isomorphy assuming only that $\mathfrak{G}'$ and $\mathfrak{H}'$ are serving (or pure) subgroups of $\mathfrak{H}$ and $\mathfrak{G}$ respectively.

C. C. Faith (University Park, Penn.).

Bandyopadhyay, Shyama Prasad. On the lattice of subgroups of finite groups. Bull. Calcutta Math. Soc. **48** (1956), 121–128.

Let C_n be the cyclic group of order n. If G has order n and if $L(G)=L(C_n)$, then $G=C_n$. Let $D^{(n)}$ be the dihedral group of order 2^n: $A^f=b^2=I$, $bab^{-1}=a^{-1}$, $f=2^{n-1}$, $n\geqq 3$. If $L(G)=L(D^{(n)})$ and if G has order 2^n, then $G=D^{(n)}$. Let $H^{(n)}$ be the quaternion group of order 2^n: $A^f=I$, $b^2=A^{f/2}$, $bab^{-1}=a^{-1}$, $n\geqq 3$. If $L(G)=L(H^{(n)})$ and if G has order 2^n, then $G=H^{(n)}$. *J. L. Brenner* (Menlo Park, Calif.).

Zacher, Giovanni. Sugli elementi modulari di un gruppo finito. Rend. Sem. Mat. Univ. Padova **26** (1956), 70–84.

A subgroup M of a group G is called a modular element of G if, for every modular sublattice R of the lattice of subgroups of G, the lattice generated by R and M is still modular. Necessary and sufficient conditions are given for M to be a modular element of G for the case where G is finite and the order of M is prime to its index in G. The author also determines the modular elements of a finite group whose Sylow subgroups are all cyclic.

P. Hall (Cambridge, England).

Neumann, B. H.; and Shepperd, J. A. H. Finite extensions of fully ordered groups. Proc. Roy. Soc. London. Ser. A. **239** (1957), 320–327.

Let G be a group and H a subgroup of G. If G is an ordered group, then the restriction to H of the order relation in G makes H also an ordered group. The authors here consider, conversely, the problem of extending a given order relation of an ordered subgroup H in G to the whole group G so that G becomes also an ordered group. The main theorem of the paper is as follows. Let G be a group having no element of finite order except 1 and let H be a normal subgroup of finite index in G. Suppose that H is a fully (linearly) ordered group and that the inner automorphisms of G induce order automorphisms of H. Then the order relation in H can be uniquely extended to the whole group G so that G becomes again a fully ordered group. Generalizations of this main result are also discussed. *K. Iwasawa* (Cambridge, Mass.).

Paige, Lowell J. A note on the Mathieu groups. Canad. J. Math. **9** (1957), 15–18.

An irreducible representation of the Mathieu group M_{23} by 11×11 matrices over the finite field F of 2 elements is given. A crucial point in the construction is to show that every set of 6 distinct columns in a certain 11×23 matrix over F is linearly independent, which was, the paper says, verified by means of the computer SWAC. It follows that in the 23-vector space V over F the subspace R orthogonal to the subspace spanned by the 11 rows of the matrix forms an exact 3-covering, i.e., for every $y\in V$ there is $x\in R$ with $d(x,y)\leqq 3$ and for every pair x, y of distinct elements of R we have $d(x,y)>6$, where the distance d of two vectors x, y is defined to be the number of coordinates i with $x_i\neq y_i$. R turns out to be the space of the representation looked for. *T. Nakayama.*

Freudenthal, Hans. Explizite Spindarstellung der Drehgruppe. Nederl. Akad. Wetensch. Proc. Ser. A. **59**= Indag. Math. **18** (1956), 515–522.

If G is the simply connected covering of the rotation group of n-space R_n, and $l=[\frac{1}{2}n]$, $\delta=n-2l=0$ (n even) or $=1$ (n odd), then G is represented in R_n by a 2-1 homomorphism ϑ, and in R_M, $M=2^l$, by a 1-1 homomorphism σ. The character χ of σ for an $x\in G$ with roots $\pm 2\omega_1, \cdots, \pm 2\omega_l$ is $\sum \exp(\pm\omega_1\pm\cdots\pm\omega_l)$, and if $\zeta(x)=\chi(\vartheta(x))$, then $\zeta(x)=\pm\sqrt{[2^{-\delta}\det(1+\vartheta(x))]}$.

Let $\mathfrak{L}$ be the linear space generated by the functions $\zeta(xa)$, $a\in G$, and N the set of all 2^{n-1} subsets of $1, \cdots, n$ with an even number of elements. If $\nu\in N$, let e_ν be one of the two elements of G for which $\vartheta(e_\nu)=$diagonal

matrix $(\lambda_1, \cdots, \lambda_n)$, $\lambda_s=1$ for $s \notin \nu$, $\lambda_s=-1$ for $s \in \nu$; $e_\mu e_\nu = \pm e_\rho$, if $\rho=(\mu\cup\nu)\setminus(\mu\cap\nu)$, $\zeta(e_\nu)=0$ if ν is not empty, $\zeta(e_\nu)=2^l\varepsilon_\nu$ $(\varepsilon_\nu=\pm 1)$ for ν empty. Then an explicit representation of $\sigma(x)$ is $2^{-l}\sum \varepsilon_\nu\zeta(xe_\nu)\sigma(e_\nu)$. For the $\sigma(e_\nu)$ the expression $\sigma(e_\nu)=\pm\prod p_s$, $s \in \nu$, is derived, where p_s is a linear mapping of R_M on itself: $\sigma(\mathrm{K}_s x)=p_s\sigma(x)p_s^{-1}$. Here K_s is an automorphism of G: $\vartheta(\mathrm{K}_s x)=T_s\vartheta(x)T_s^{-1}$, T_s diagonal matrix of R_n with -1 on the sth place and $+1$ on all other places.

For the p_s the equation $p_sp_t+p_tp_s=0$ $(s\neq t)$ holds, they generate a linear space P, and -1, p_s a group F_n of 2^{n-1} elements in R_M. The algebra $A(F_n)$ is described, so that the $\sigma(e_\nu)$ can be explicitly determined. It is further shown how the transition from σ to ϑ can be made. This paper thus gives information complementary to that found in E. Cartan [Bull. Soc. Math. France **41** (1913), 53–96, esp. p. 91=Oeuvres complètes, partie 1, v. 1, Gauthier-Villars, Paris, 1952, pp. 355–398, esp. p. 392; MR **14**, 393] and R. Brauer and H. Weyl [Amer. J. Math. **57** (1935), 425–449]; reference is also made to a paper by B. A. Rozenfel'd [Trudy Sem. Vektor. Tenzor. Anal. **6** (1948), 506–514; MR **14**, 797]. *D. J. Struik.*

McLachlan, Dan, Jr. Symmetry in reciprocal space. Acta Cryst. **9** (1956), 318.

The author describes a procedure by which each two-dimensional point group yields a three-dimensional point group including the central inversion. When the two-dimensional group already contains the central inversion, no change is needed except to re-interpret reflections in lines as reflections in planes, and so on. In the remaining cases the procedure is to adjoin the inversion in a centre outside the plane. In a similar manner, three-dimensional point groups yield four-dimensional point groups.
H. S. M. Coxeter (Toronto, Ont.).

Thierrin, Gabriel. Sur les automorphismes intérieurs d'un demi-groupe réductif. Comment. Math. Helv. **31** (1956), 145–151.

Un demi-groupe D est réductif à droite si la relation $ax=bx$ pour tout $x \in D$ entraîne $a=b$. Un complexe $H\subseteq D$ est réducteur à droite si $ah=bh$ pour tout $h \in H$ entraîne $a=b$. On a les définitions symétriques. D est réductif s'il est réductif à droite et à gauche, H est un complexe réducteur s'il est réducteur à droite et à gauche. Le complexe H est dit r-intérieur 1) s'il est réducteur et 2) si pour tout $a \in D$ il existe $b, c \in D$ tels que l'on ait $ha=bh$ et $ah=hc$ pour tout $h \in D$.

Soit $\mathfrak{F}$ l'ensemble des complexes r-intérieur de D. Soit $\mathfrak{F}\neq\emptyset$ et $H \in \mathfrak{F}$. La corréspondance $a\to b$ définie par $ha=bh$ pour tout $h \in H$ est un automorphisme α_H de D appellé automorphisme intérieur de première catégorie. L'application inverse $b\to a$ est appelée automorphisme intérieur de deuxième catégorie.

Soit I_1 (I_2) l'ensemble des automorphismes intérieurs de première (deuxième) catégorie de D. I_1 est un semi-groupe homomorphe au demi-groupe $\mathfrak{F}$. I_1 est un groupe si et seulement si $I_1=I_2$. Tout demi-groupe T peut être plongé dans un demi-groupe réductif D tel que tout automorphisme de T soit induit sur T par un automorphisme intérieur de première catégorie de D.

Soit D un demi-groupe réductif, $\mathfrak{F}\neq\emptyset$. L'élement $b \in D$ est dit conjugué à droite d'un élément a s'il existe $H \in \mathfrak{F}$ tel que l'on ait $\alpha_H(a)=b$. La relation de conjugaison σ est définie comme suit: $a\sigma b \leftrightarrow b$ est conjugué à droite de a. La condition nécessaire et suffisante pour que σ soit une relation d'équivalence est donnée. Soit X un complexe de D. La relation ρ_X d'équiconjugaison à droite est définie dans $\mathfrak{F}$ par $H\rho_X K \leftrightarrow \alpha_H(X)=\alpha_K(X)$. Les propriétés de ρ_X sont étudiées.

L'auteur propose d'étendre les résultats de P. Dubreil [Mém. Acad. Sci. Inst. France **63** (1941), no. 3; MR **8**, 15] et R. Croisot [Bull. Soc. Math. France **82** (1954), 161–194; MR **16**, 215] concernant les automorphismes intérieurs de semi-groupes à une classe plus générale de demi-groupes. Il n'est pas surprenant que quelques unes des notions introduites semblent considérablement artificielles.
Š. Schwarz (Bratislava).

Iséki, Kioyoshi. Contributions to the theory of semi-groups. V. Proc. Japan Acad. **32** (1956), 560–561.

[For parts I–IV see same Proc. **32** (1956), 174–175, 225–227, 323–324, 430–435; MR **17**, 1184; **18**, 282.] Let S be a finite commutative semigroup. The reviewer introduced a disjoint conjugate class decomposition of S and showed that any character on S takes the same value on each conjugate class [Czechoslovak Math. J. **4(79)** (1954), 291–295; MR **16**, 1086]. The author extends the existence of such decompositions to some more general types of semigroups. *Š. Schwarz* (Bratislava).

Devidé, Vladimir. Über eine Klasse von Gruppoiden. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **10** (1955), 265–286. (Serbo-Croatian summary)

In dieser Arbeit werden Gruppoide, deren Axiome gewisse Abschwächungen der Gruppenaxiome darstellen, untersucht. Der Grundgedanke besteht darin, die durch die Gruppenaxiome ausgedrückten Forderungen nicht an die Gruppoidenelemente selbst sondern an gewisse schlichte Funktionen derselben zu stellen. Es sei S ein Gruppoid. Mit $a, b, c, m, \bar{a}, \cdots$ werden Elemente von S und mit $\varphi, \psi, \chi, \lambda, \cdots$ schlichte Abbildungen des Gruppoids auf sich bezeichnet. Das Gruppoid S wird "C-Gruppe" genannt, wenn 1) $(\exists\varphi, \psi, \chi)(\forall\, a, b, c)ab\cdot c=\varphi a(\psi b\cdot\chi c)$, 2) $(\exists\lambda)(\exists m)(\forall a)$ $ma=\lambda a$, 3) $(\forall a)(\exists\bar{a})$ $\bar{a}a=m$ gilt. Als unmittelbare Folgerungen dieser Axiome werden die linksseitige Kürzungsregel, die eindeutige Lösbarkeit bzw. Lösbarkeit der Gleichung $ax=b$ bzw. $ya=b$ festgestellt. Wird zu den Axiomen 4) $(\exists n)(\exists\delta)(\forall a)$ $an=\delta a$ hinzugefügt, so entsteht eine "D-Gruppe", in der die eindeutige Lösbarkeit der Gleichung $ya=b$, die rechtsseitige Kürzungsregel, sowie weitere Erscheinungen auftreten. Ferner werden vom Verf. die sog. "A-Gruppen", d.h. die mit Hilfe von Automorphismen φ, ψ, χ definierten D-Gruppen, eingehend untersucht. — Die Arbeit zeigt neue Wege zu einer über der allgemeinen Gruppoidenlehre stehenden Approximation der Gruppentheorie.
O. Borůvka (Brno).

Sagastume Berra, A. E. Fundamental homomorphism theorems for groupoids. Rev. Un. Mat. Argentina **17** (1955), 205–212 (1956). (Spanish)

A set G closed with respect to binary associative multiplication and having a unit element 1 is called a groupoid. (A zero element 0 is also postulated but not in fact used.) A subgroupoid N containing 1 is called normal if for all x of G the sets xN, Nx coincide. Then G is homomorphic to the factor groupoid G/N which consists of all the sets xN under the obviously defined multiplication; these sets may overlap. The homomorphism is denoted $H(N)$.

Given a homomorphism H of G onto another groupoid Γ, the sets of elements of G which are inverse images of the elements of Γ form under an obviously defined multi-

plication a homomorph of G, denoted G/H, isomorphic to Γ. The inverse image of the unit element of Γ is a subgroupoid of G called the nucleus of H, denoted $E(H)$.

If N is a normal subgroupoid of G, then $E(H(N))=N$. On the other hand, even when $E(H)$ is normal, we may have $H(E(H))\neq H$. If $N\subseteqq G'\subseteqq G$, N and G' being normal subgroupoids of G, then G'/N is a normal subgroupoid of G/N, and $(G/N)/(G'/N)$ is isomorphic to G/G'.
I. M. H. Etherington (Edinburgh).

See also: Ohkuma, p. 868; Ghermănescu, p. 944; Dolginov, p. 973.

THEORY OF NUMBERS

General Theory of Numbers

Kanold, Hans-Joachim. Über die Verteilung der vollkommenen Zahlen und allgemeinerer Zahlenmengen. Math. Ann. **132** (1957), 442–450.

In a preceding paper [Math. Ann. **132** (1956), 246–255; MR **18**, 718] the author generalized a theorem of L. E. Dickson. In the first parts of the present paper some of his results are improved and extended. Their statements are too involved to be quoted explicitly. Let $f(n)$ be an arithmetical function satisfying certain conditions. Let $A(x)=A(x, c)$ denote the number of solutions n of $f(n)=c$, $n\leqq x$ (c need not be an integer). Then $A(x)$ and related counting functions are estimated. In particular the cases $f(n)=n^{-r}\sum_{d|n} d^r$ and $f(n)=\prod_{p|n}(1-p^{-r})$ are discussed. — In the last section the author shows that the number $A(x)$ of perfect numbers $\leqq x$ is
$$O\left(x^{\frac14}\frac{\ln x}{\ln\ln x}\right).$$
This improves the author's previous estimate $A(x)=O(\sqrt{x})$ [ibid. **131** (1956), 390–392; MR **18**, 16]. *P. Scherk.*

Kanold, Hans-Joachim. Über mehrfach vollkommene Zahlen. II. J. Reine Angew. Math. **197** (1957), 82–96.

[For part I see same J. **194** (1955), 218–220; MR **17**, 238.] Italics denote positive integers. Let $n=\prod_1^k p_r^{\alpha_r}$ be the factorization of n into powers of mutually distinct prime numbers. Assume that $s=n^{-1}\sum_{d|n} d$ is an integer.

Theorems 1–3: Suppose t is a common divisor of all the numbers α_r+1 with not more than one exception; $k>2$. Then $t|sn$. If in addition t or s are powers of 2, then $t^3|n$. — Theorem 4: There is no odd $n<10^{20}$. *P. Scherk.*

Christofferson, Stig. Über eine Klasse von kubischen diophantischen Gleichungen mit drei Unbekannten. Ark. Mat. **3** (1957), 355–364.

The author considers the Diophantine equation
$$(*)\qquad u^3+ab^2v^3+a^2bw^3-3abuvw=c$$
in integers u, v, w when the integers a, b, c are given. This may be regarded as an equation $\mathrm{Norm}(u+v\alpha+w\beta)=c$ where $\alpha=(ab^2)^{\frac13}>0$ and $\beta=(a^2b)^{\frac13}>0$ are numbers of the same cubic field. In the first part it is shown, by a direct modification of a method of Dirichlet, that when $c=1$ there is always a solution in which, u, v, w are less than a rather elaborate bound which is given explicitly in terms of a, b. In the second part the author shows that if c is positive and $u+v\alpha+w\beta>c^{\frac13}$ then $u\geqq 0$, $v\geqq 0$, $w\geqq 0$ with strict inequality if either $c\geqq 16$ or $u+v\alpha+w\beta\geqq(4c)^{\frac13}$. An example is given with $c=15$, $u+v\alpha+w\beta>c^{\frac13}$ but $uvw=0$. In the third part it is shown that if $\varepsilon>1$ is the fundamental unit of the ring of $u+v\alpha+w\beta$ then to every solution of (*) there is an 'associated' solution in which u, $v\alpha$, $w\beta$ lie between the bounds $-\frac13(2-\varepsilon^{-1})(c\varepsilon)^{\frac13}$ and $(c\varepsilon)^{\frac13}$. This is better than the direct application of the well-known bounds of Hecke [Vorlesungen über die Theorie der algebraischen Zahlen, 2. Aufl., Akademische Verlagsgesellschaft, Leipzig, 1954; MR **16**, 571] and Nagell [Math. Z. **34** (1931), 183–193, p. 191] but is obtained by an immediate modification of their general method applied to this particular case. *J. W. S. Cassels.*

Stolt, B. A note on triangular numbers. Portugal. Math. **15** (1956), 87–88.

An elementary proof of the following theorem: Let n be an integer expressed in the form $n=9k+r$, with $2<r<12$. Then n can be expressed as the sum of r triangular numbers of the form Δ_{3s+1}. *L. Moser.*

See also: Siegel, p. 873; Cohen, p. 875; Carlitz, p. 874; Hughes, p. 921.

Analytic Theory of Numbers

Siegel, Carl Ludwig. A generalization of the Epstein zeta function. J. Indian Math. Soc. (N.S.) **20** (1956), 1–10.

Let $\mathfrak{x}'\mathfrak{S}\mathfrak{x}=\mathfrak{S}[\mathfrak{x}]$ be an even quadratic form of m variables, with signature n, r $(n+r=m)$. Let $\mathfrak{a}$ be a vector such that $\mathfrak{S}\mathfrak{a}$ is integral, and $d=(-1)^r|\det\mathfrak{S}|$. The Gaussian sum $g_{\rho\mathfrak{a}}$ is defined by
$$g_{\rho\mathfrak{a}}=\textstyle\sum_{\mathfrak{x}\bmod\gamma}\exp(\pi i\rho\mathfrak{S}[\mathfrak{x}+\mathfrak{a}]),$$
if α and γ are integers, $(\alpha,\gamma)=1$, $\gamma>0$, $\rho=\alpha/\gamma$. Furthermore, the number $g_{\mathfrak{a}}$ is defined as 1 if $\mathfrak{a}$ is an integral vector, and 0 otherwise. The author introduces the function
$$\varphi_{\mathfrak{a}}(s)=$$
$$g_{\mathfrak{a}}+e^{\frac14\pi i(n-r)}d^{-\frac12}\textstyle\sum_\rho g_{\rho\mathfrak{a}}\gamma^{-1-s}(z-\rho)^{\frac12(r-1-s)}(\bar z-\rho)^{\frac12(n-1-s)},$$
where ρ runs through all rational numbers; $z=\xi+i\eta$ and $s=\sigma+it$ are complex numbers, with $\eta>0$, $\sigma>1+\frac12 m$. The author proves that $\varphi_{\mathfrak{a}}(s)$ is (by analytic continuation) a meromorphic function of s. And, if $\mathfrak{S}$ is a stem-form, it satisfies the functional equation $\varphi_{\mathfrak{a}}(s)=\eta^{\frac12 m-s}q(s)\varphi_{\mathfrak{a}}(m-s)$, where $q(s)=$
$$\pi d^{-\frac12}2^{1-s+\frac12 m}\Gamma(s-\tfrac12 m)\Gamma^{-1}(\tfrac12(s+1-n))\Gamma^{-1}(\tfrac12(s+1-r))f(s),$$
$f(s)=\sum g_{\rho 0}\gamma^{-1-s}$
(ρ runs through the rationals in $0\leqq\rho<1$) in a function of the singular series type.

In the proof, the author uses the transformation formula (under modular substitutions of z) of the theta series
$$f_{\mathfrak{a}}(z,\mathfrak{w})=\textstyle\sum_{\mathfrak{x}}\exp(\pi i\mathfrak{R}[\mathfrak{x}+\mathfrak{a}]+2i\mathfrak{w}'(\mathfrak{x}+\mathfrak{a})).$$
Here $\mathfrak{w}$ is a real vector, $\mathfrak{x}$ runs through all integral vectors, $\mathfrak{P}$ is a majorant of $\mathfrak{S}$ (i.e. $\mathfrak{P}\mathfrak{S}^{-1}\mathfrak{P}=\mathfrak{P}$, $\mathfrak{P}=\mathfrak{P}'>0$), and $\mathfrak{R}=\xi\mathfrak{S}+i\eta\mathfrak{P}$. The special case $\mathfrak{w}=0$ occurs in a previous paper [Math. Ann. **124** (1951), 17–54; MR **16**, 800]. A theta series vector is considered: if $\mathfrak{a}$ runs through a

complete set of representatives $\mathfrak{a}_1, \cdots, \mathfrak{a}_l$ (mod 1) (all satisfying $\mathfrak{S}\mathfrak{a}\equiv 0$ (mod 1)), then $f_{\mathfrak{a}}(z, \mathfrak{w})$ describes a column vector of length l. Under modular substitutions of z this column is multiplied on the left by an $l\times l$ matrix. The coefficients of this matrix are closely related to the function $\varphi_{\mathfrak{a}}(s)$. *N. G. de Bruijn* (Amsterdam).

Glatfeld, Martin. Anwendungen der wienerschen Methode auf den Primzahlsatz für die arithmetische Progression. J. London Math. Soc. 32 (1957), 67–73.

The author applies N. Wiener's general theory of Tauberian theorems to the proof of the prime number theorem for arithmetical progressions:

$$(1)\qquad \pi(x;k,l)=\sum_{\substack{p\leq x\\ p\equiv l(k)}}1=\frac{1}{\varphi(k)}\frac{x}{\log x}+o\left(\frac{x}{\log x}\right),$$

where k, l are fixed integers, $k>0$, $(k,l)=1$. The fundamental theorem of Wiener's theory, in the form given to it by H. R. Pitt, is applied directly to the proof of asymptotic formulae for the sums

$$\sum_{n\leq u}\frac{\chi(n)\mu(n)}{n},\quad \sum_{n\leq u}\chi(n)\mu(n),$$

where $\chi(n)$ is a character (mod k) and $\mu(n)$ is the Möbius function; and the truth of (1) is then inferred from equivalence theorems of H. N. Shapiro [Comm. Pure Appl. Math. 2 (1949), 293–308; MR 11, 419]. [For a similar treatment of the case $k=1$ see G. H. Hardy, Divergent series, Oxford, 1949, pp. 303–304, 378–380; MR 11, 25]. The formal relation of this proof to proofs based on Lambert's series is discussed. *A. E. Ingham.*

Halberstam, H. An asymptotic formula in the theory of numbers. Trans. Amer. Math. Soc. 84 (1957), 338–351.

The formula in question is

$$\sum_{\nu=1}^{n-1}\sigma_\alpha(\nu)\sigma_\beta(n-\nu)=A_1\sigma_{\alpha+\beta+1}(n)+A_2n^\alpha\sigma_{-\alpha+\beta+1}(n)+$$
$$A_3n^\beta\sigma_{\alpha-\beta+1}(n)+A_4n^{\alpha+\beta}\sigma_{-\alpha-\beta+1}(n)+O(n^\omega\log^\lambda n),$$

where $\sigma_r(m)$ is the sum of the rth powers of the (positive) divisors of the positive integer m, and A_1, A_2, A_3, A_4, ω, λ are constants depending on the positive constants α, β. This is an improvement of previous results, stated by the reviewer and proved by the author, involving the first explicit term and weaker error terms. The improvement affects the case $\alpha+\beta<1$ and amounts in the main to a reduction of ω from $\max(\alpha,\beta)+1$ to $\frac{3}{4}+\frac{1}{2}(\alpha+\beta)$ or $\frac{1}{2}+(\alpha+\beta)$ according as $\alpha+\beta<\frac{1}{2}$ or $\geq\frac{1}{2}$. This is achieved by the use, in place of the original elementary method, of an analytical method (the Hardy-Ramanujan-Littlewood 'circle method') on the lines of Estermann's treatment of the corresponding problem with $\alpha=\beta=0$ [Proc. London Math. Soc. (2) 31 (1930), 123–133]. The present method uses A. Weil's estimate [Proc. Nat. Acad. Sci. U.S.A. 34 (1948), 204–207; MR 10, 234]) of Kloosterman sums. Estermann used earlier estimates of these sums.

A. E. Ingham (Cambridge, England).

Carlitz, L. Note on sums of four and six squares. Proc. Amer. Math. Soc. 8 (1957), 120–124.

Using the formula $\wp'(u)=-\sigma(2u)/\sigma^4(u)$ and the elementary methods of a previous note [Nieuw Arch. Wisk. (3) 3 (1955), 129–133; MR 17, 460] the author gives short proofs of familiar formulas for the number of representations of an integer as a sum of four and six squares, also of the formula $\theta_3{}^4(g)=\theta_0{}^4(g)+\theta_2{}^4(g)$. *N. J. Fine.*

Kasch, Friedrich. Abschätzung der Dichte von Summenmengen. III. Math. Z. 66 (1956), 164–172.

Let A, B, $\cdots$ denote sets of non-negative integers. The set $C=A+B$ consists of those integers c which permit representations $c=a+b$, $a\subset A$, $b\subset B$. Let $A(k)$, $\cdots$ denote the number of positive elements $\leq k$ of A, $\cdots$. Let $0\subset B$, $1\subset B$ and let $g(m)$ denote the smallest number of elements of B whose sum equals m. Put

$$\alpha=\inf_{k=1,\cdots,n}A(k)/k \text{ and } \lambda=\sup_{k=1,\cdots,n}k^{-1}\sum_1^k g(m).$$

A. Brauer proved: Let $\alpha\leq\frac{1}{2}$, $\lambda\geq\frac{3}{2}$. Then

$$(1)\qquad c(n)/n\geq\alpha+\tfrac{1}{2}\{\lambda-\sqrt{(\lambda^2-3\alpha(1-\alpha))}\}$$

[Math. Z. 63 (1956), 529–541; MR 17, 712]. Utilizing an idea of Brauer's and his own work [ibid. 64 (1956), 243–257; MR 17, 712], the author now proves (1) under the weaker assumption

$$\lambda\geq\frac{3+\alpha(1-\alpha)(3-2\alpha)^2}{2(3-2\alpha)}.$$

He also proves the stronger estimate

$$\frac{c(n)}{n}\geq\alpha+\frac{1}{2}\left(\lambda-\frac{\alpha}{4}-\sqrt{\left(\left(\lambda-\frac{\alpha}{4}\right)^2-3\alpha(1-\alpha)\right)}\right)$$

for

$$\lambda\geq\frac{6+39\alpha-38\alpha^2+8\alpha^3}{2(3-2\alpha)(2+3\alpha-2\alpha^2)}.$$

Similar results are obtained for asymptotic densities.

P. Scherk (Saskatoon, Sask.).

See also: Carlitz, p. 875; Sierpiński, p. 888.

Theory of Algebraic Numbers

Linnik, Yu. V. More on the analogues of the ergodic theorems for the imaginary quadratic field. Dokl. Akad. Nauk SSSR (N.S.) 109 (1956), 694–696. (Russian)

Let $\mathfrak{H}$ be the group of properly primitive positive definite binary quadratic forms $(a,b,c)=ax^2+2bxy+cy^2$ of odd determinant $D=ac-b^2$. Each such form corresponds to a 'fundamental point' $\mathfrak{a}=(a_0,b_0,c_0)$ ($a_0=a/\sqrt{D}$, etc.) on the hyperboloid $a_0c_0-b_0{}^2=1$ ($a_0>0$), and the reduced forms correspond to points inside a triangular region Δ_0 whose Lobatchesvkian area $\mu(\Delta_0)$ is taken to be 1. Let p be an odd prime such that $(-D/p)=1$ and let $\mathfrak{p}=(a_0', b_0', c_0')$ denote the fundamental point corresponding to either of the forms $(p,\pm\xi,n)$ of $\mathfrak{H}$. By Gaussian composition we obtain a third fundamental point $\mathfrak{a}\mathfrak{p}=(a_0'', b_0'', c_0'')$, and so a transformation $\mathfrak{T}$ is defined which transforms the class of forms $\mathfrak{a}$ into the class $\mathfrak{a}\mathfrak{p}$; powers of this transformation can then be defined Let Ω be a simply-connected subset of Δ_0 bounded by piece-wise smooth arcs and of Lobatchevskian area $\mu(\Omega)$ and let $f_\Omega(P)=1$ if $P\in\Omega$ and $f_\Omega(P)=0$ if $P\notin\Omega$. The author states, without proof, the following theorem concerning the behaviour of the ergodic mean value of $f_\Omega(\mathfrak{a}\mathfrak{T}^t)$:

$$\frac{1}{s}\{f_\Omega(\mathfrak{a})+f_\Omega(\mathfrak{a}\mathfrak{T})+\cdots f_\Omega(\mathfrak{a}\mathfrak{T}^{s-1})\}=\mu(\Omega)\{1+o(1)\},$$

for $s > c_0 \log D$, as $D \to \infty$, for all classes $\mathfrak{a}$ with the possible exception of $o\{h(-D)\}$ classes. Here $h(-D)$ is the order of $\mathfrak{H}$. In this result $\mathfrak{T}$ can be replaced by $\mathfrak{T}^r$ and the positive constant c_0 will then depend on r. Various consequences of this theorem are stated. In particular, if $\mathfrak{G}$ is a subgroup of $\mathfrak{H}$ of bounded index g and r is chosen so that $\mathfrak{p}^r \in \mathfrak{G}$, then the number of points in any coset of $\mathfrak{G}$ which are mapped into Ω by the transformation $\mathfrak{T}^r$ is

$$\frac{h(-D)}{g}\mu(\Omega)\{1+o(1)\}$$

as $D \to \infty$. If $\mathfrak{G}=\mathfrak{H}$ this yields an earlier result of the author's [Vestnik Leningrad. Univ. **10** (1955), no. 2, 3–23, no. 5, 3–32, no. 8, 15–27; MR **18**, 193].

R. A. Rankin (Glasgow).

Cohn, Harvey. Some algebraic number theory estimates based on the Dedekind eta-function. Amer. J. Math. **78** (1956), 791–796.

Using results of Dedekind on his eta-function, the author obtains surprising improvements of well-known estimates of Landau [Göttinger Nachrichten, Math. Phys. Klasse, 1918, p. 95] for the class-number h, and for the product of the class-number with the logarithm of the fundamental unit $\varepsilon > 1$, of a so-called pure cubic field. These are

(1) $\qquad h \log \varepsilon = O(|d|^{\frac{1}{2}} \log |d| \log\log |d|)$,

and this easily implies

(2) $\qquad h = O(|d|^{\frac{1}{2}} \log\log |d|)$,

where d is the field discriminant. For a cubic field Landau had $h = O(|d|^{\frac{1}{2}} \log^2 |d|)$. The reference to Dedekind is Ges. math. Werke. Vol. 2, Braunschweig, 1931.

S. Chowla (Boulder, Colo.).

Cohen, Eckford. Congruences in algebraic number fields involving sums of similar powers. Trans. Amer. Math. Soc. **83** (1956), 547–556.

Let $Q_s(\rho)$ denote the number of solutions of the congruence $\alpha_1 X_1^m + \cdots + \alpha_s X_s^m + \rho \equiv 0 \pmod{p^\lambda}$, where p is a prime ideal of a finite extension F of the rational field and $\alpha_1, \alpha_2, \cdots, \alpha_s, \rho$ are integers of F. Generalizing results which the author has obtained earlier in the case $s=2$ he gives exact formulas for $Q_s(\rho)$ involving the generalized Jacobi sums. In the general case these formulas are complicated but can be simplified in special cases.

H. Bergström (Göteborg).

Carlitz, L. Weighted quadratic partitions over GF$[q, x]$. Duke Math. J. **23** (1956), 493–505.

Let GF(q) be the Galois field with $q = p^n$ elements (p prime, $p > 2$). In an earlier paper [Canad. J. Math. **5** (1953), 317–323; MR **15**, 508] the author used the function $e_0(a) = \exp\,(2\pi i p^{-1} \sum_{\nu=1}^{n-1} a^{p^\nu})$ and considered the sum $\sum e_0(2b_1x_1 + \cdots + 2b_rx_r)$ over all $x_i \in \mathrm{GF}(q)$ such that $a_1x_1^2 + \cdots + a_rx^2 = a$ (a_i, $a \in \mathrm{GF}(q)$). This sum could be evaluated explicitly or expressed in terms of Kloosterman sums. Now he considers the more general case when $a, x_1, \cdots, x_n$ are polynomials of variable x with coefficients in GF(q). Then introducing $b_1, \cdots, b_r$ as elements of the power series field consisting of the quantities $\sum_{-\infty}^{m} c_j x^j$, $c_j \in \mathrm{GF}(q)$, he considers the corresponding sums, where now $e_0(a)$ is replaced by a function $e(\xi)$,

$$e(\xi) = \exp\frac{2\pi i}{p}\sum_{\nu=1}^{n-1} c_{-1}^{p^\nu} \text{ for } \xi = \sum_{-\infty}^{m} c_j x^j.$$

Also in this general case the sum can be given by the help of Kloosterman sums and related sums.

H. Bergström (Göteborg).

See also: Petersson, p. 867; Christofferson, p. 873.

Geometry of Numbers

Cassels, J. W. S. On a result of Marshall Hall. Mathematika **3** (1956), 109–110.

It has been shown by the reviewer [Ann. of Math. (2) **48** (1947), 966–993; MR **9**, 226] that any real number is the sum (or difference) of two continued fractions with partial quotients at most 4. Given any two real numbers β_1 and β_2, let γ_1 and γ_2 be continued fractions with partial quotients at most 4 such that $\gamma_1 - \gamma_2 = \beta_1 - \beta_2$. Then with $\alpha = \gamma_1 - \beta_1 = \gamma_2 - \beta_2$ we will have $x|(\alpha+\beta_1)x - y| > C$ and $x|(\alpha+\beta_2)x - y| > C$ for all integers $x > 0$ and y, where we may take $C = 1/5$. This result is generalized here to any number of real numbers β_i. It is shown that if $\beta_1, \cdots, \beta_r$ are any real numbers, then there exists a real number α such that $x|(\alpha+\beta_j)x - y| > C_r$ ($j = 1, \cdots, r$) for all integers $x > 0$ and y, where we can take $C_r = \frac{1}{8}(r+1)^2$.

Marshall Hall, Jr. (Columbus, Ohio).

Postnikov, A. G. Properties of solutions of Diophantine inequalities in the field of formal power series. Mat. Sb. N.S. **40(82)** (1956), 295–302. (Russian)

The author proves the results already announced and some similar but simpler ones [Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 21–22; MR **18**, 22].

J. W. S. Cassels (Cambridge, England).

Ehrhart, Eugène. Sur l'empilement réticulaire d'ovales ou d'ovoïdes. C. R. Acad. Sci. Paris **244** (1957), 550–553.

The author gives simple geometrical proofs of results of Minkowski reducing the problem of finding the closest lattice packing of a given convex domain or body to the corresponding problem for its difference domain or body.

C. A. Rogers (Birmingham).

Müller, Claus. Die Grundprobleme der Geometrie der Zahlen. Math.-Phys. Semesterber. **5** (1956), 63–70.

This article is mainly expository. It contains two proofs of Minkowski's Fundamental Theorem, one of which appears to be a new variant of C. L. Siegel's analytical proof [Acta Math. **65** (1935), 307–323], and a discussion of the number of lattice points in a large circle.

C. A. Rogers (Birmingham).

Cugiani, Marco. Sugli insiemi numerici del tipo $p^n - q^n a$. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **90** (1956), 209–220.

In a previous paper [Boll. Un. Mat. Ital. (3) **10** (1955), 489–497; MR **17**, 829] the author studied the set of points whose abscissas are of the form $p^2 - q^2\alpha$, where α is a fixed real number and p, q range over the integers. He showed that for almost all α this set has 0 for an accumulation point. The present paper deals with the set $I_n(\alpha)$ of points of the form $p^n - q^n\alpha$ where n is an integer $\geqq 3$. In this case $I_n(\alpha)$ has no finite accumulation point for almost all α. In particular this is true when α is any algebraic number, a fact which follows from the recent theorem of Roth [Mathematika **2** (1955), 1–20, 168; MR **17**, 242].

Other results for even values of $n\geq 4$ show that the derivative of $I_n(\alpha)$ can be either of the two half-axes or a proper subset thereof, depending on the choice of α.
D. H. Lehmer (Berkeley Calif.).

ANALYSIS

✱ Фихтенгольц, Г. М. [Fihtengol'c, G. M.] Основы математического Анализа. Том I. [Foundations of mathematical analysis. Vol. I.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 440 pp. 9.75 rubles.

Contains fundamental notions including Taylor's formula but no general theory of series which is promised for a second volume. Combines painstaking rigor with clarity. Presentation is accompanied by historical remarks; last chapter gives a very careful sketch (23 pages) of historical origins of fundamental concepts of analysis. Present edition 50000.

Functions of Real Variables

Pettineo, Benedetto. Sur la dérivabilité des fonctions. C. R. Acad. Sci. Paris **243** (1956), 553–554.

Let the real function $F(x)$ be continuous in $[a, b]$ and let $C\subset(a, b)$ be a closed set. Cover C by a finite system S of non-overlapping subintervals $[\alpha_\gamma, \beta_\gamma]$ of (a, b). Then F has the variation $\nu[C]$ on C if for every $\varepsilon>0$ there is a $\delta>0$ such that $|\nu[C]-\sum_\gamma (F(\beta_\gamma)-F(\alpha_\gamma))|<\varepsilon$ for each system S for which all $|\beta_\gamma-\alpha_\gamma|<\delta$. Set $C^x=[a, x]\cap C$. Then F is absolutely continuous on C if $\nu[C^x]$ is absolutely continuous and, moreover, the author calls F "almost absolutely continuous" ["presque absolument continue"] in $[a, b]$ if for every $\varepsilon>0$ there is a closed set $C\subset(a, b)$ with $(b-a)-m(C)<\varepsilon$ on which F is absolutely continuous. The main result of this note is the theorem: In order that F have a finite derivative almost everywhere in $[a, b]$, it is necessary and sufficient that F be almost absolutely continuous in $[a, b]$.
A. Rosenthal (Lafayette, Ind.).

Marcus, S. Sur les fonctions de Hamel. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **8** (1956), 517–528. (Romanian. Russian and French summaries)

Marcus, S. Contribution à une analyse des fonctions réelles, basée sur la notion de catégorie (au sens de Baire). Acad. R. P. Romîne. Stud. Cerc. Mat. **7** (1956), 251–272. (Romanian. Russian and French summaries)

Stoilow, S. Sur la convergence continue. Acad. R. P. Romîne. Stud. Cerc. Mat. **7** (1956). 247–250. (Romanian. Russian and French summaries)

Goodner, D. B. Extensions of the law of the mean. Amer. Math. Monthly **64** (1957), 185–186.

Using a theorem of Kametani [Nat. Sci. Rep. Ochanomizu Univ. **1** (1951), 1–5; MR **14**, 625] a new proof is given for the following theorem proved earlier by Vučković [Srpska Akad. Nauka. Zb. Rad. **18**, Mat. Inst. **2** (1952), 159–166; MR **14**, 625] and the reviewer [Amer. Math. Monthly **62** (1955), 178–179]. If the real valued functions f and g are continuous on the interval $a\leq x\leq b$ and if the right and left hand derivatives f_+', f_-', g_+', g_-' exist on $a<x<b$, then there exist numbers x_0, p, q with $a<x_0<b$, $p\geq 0$, $q\geq 0$, $p+q=1$ for which

$$(pf_+'(x_0)+qf_-'(x_0))(g(b)-g(a))=(pg_+'(x_0)+qg_-'(x_0))(f(b)-f(a)).$$

W. R. Utz (Columbia, Mo.).

Gołąb, S.; et Łojasiewicz, S. Un théorème sur la valeur moyenne θ dans la formule des accroissements finis. Ann. Polon. Math. **3** (1956), 118–125.

The value θ in the mean value theorem

$$f(b)-f(a)=(b-a)f'(a+\theta(b-a))$$

is considered as a function $\theta(a, b)$ of a and b. The authors solve the problem of finding all functions f (satisfying suitable conditions) for which $\theta(a, b)$ is homogeneous (of order zero). The solution of this problem is obtained by two different methods. For the first method (due to the first author) it is assumed that f''' exists and is continuous and that $f''(x)\neq 0$. Then the problem is reduced to the solution of the differential equation (*) $xf'''(x)/f''(x)=C$ which is to be satisfied except perhaps for $x=0$. For the second method weaker assumptions are made, namely f' is supposed to exist and to be strictly monotone in (a, b). Then the problem leads to a system of two linear functional equations. The proof of a lemma used therein will be published separately. Both methods give the following three families of curves as the solutions of the problem:

$$(1)\qquad f(x)=\alpha x \ln|x|+\beta x+\gamma,$$

$$(2)\qquad f(x)=\alpha \ln|x|+\beta x+\gamma,$$

$$(3)\qquad f(x)=\alpha x^\delta+\beta x+\gamma.$$

After completion of this paper, the authors observed that a formula implying (*) had already been obtained by V. Gonçalves [Portugal. Math. **2** (1941), 121–138; MR **3**, 145].
A. Rosenthal (Lafayette, Ind.).

Acel', Ya. [Aczél, J.] On the theory of means. Colloq. Math. **4** (1956), 33–55. (Russian)

Starting with a review of the early work of A. N. Kolmogoroff [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fiz. Mat. Nat. (6) **12** (1930), 388–391] on the axiomatic theory of mean-value functions, the author gives a unified development of the subject, to which he adds some new results.

An extensive bibliography, especially as regards the excellent contributions of the Polish and Hungarian schools, is appended.
E. F. Beckenbach.

Ciorănescu, N. A new mean-value formula and its application to the approximate calculation of definite integrals. Gaz. Mat. Fiz. Ser. A. **8** (1956), 19–23. (Romanian)

Continuing his work with the mean-value formula

$$\frac{M_p[fg]-M_p[f]M_p[g]}{M_p[\varphi\psi]-M_p[\varphi]M_p[\psi]}=\frac{f'(\xi)g'(\xi')}{\varphi'(\xi)\psi'(\xi')},$$

where $M_p[F]=\int_a^b p(x)F(x)dx/\int_a^b p(x)dx$, $p(x)$ is an arbitrary nonnegative weight function, and ξ and ξ' are suitable values in the open interval (a, b) [Bull. Math. Soc. Roumaine Sci. **46** (1944), 107–112; MR **7**, 418], the author considers the special case $p(x)=1$, $Q(x)=x-a$, $\psi(x)=x-b$, which leads to the formula

$$(b-a)\int_a^b f(x)g(x)dx-\int_a^b f(x)dx\int_a^b g(x)dx=\frac{(b-a)^4}{12}f'(\xi)g'(\xi').$$

The present paper is concerned with applications of this

last formula to the approximate calculation of definite integrals. Thus it is shown that

$$\int_a^b F(x)dx=\sum_{p=0}^{n-1}(-1)^p\frac{n!(2n-p-1)!}{(2n)!(n-p-1)!(p+1)!}$$
$$\times(b-a)^{p+1}[F^{(p)}(b)+(-1)^pF^{(p)}(a)]$$
$$+(-1)^p\frac{n!}{12(2n)!}(b-a)^{n+3}F^{(n+1)}(\xi)X_n'(\xi'),$$

where $X_n(x)=[(x-a)^n(x-b)^n]^{(n)}/[n!(b-a)^n]$ is the nth Legendre polynomial relative to the interval (a, b). It is shown further that if the points $x_k=k(b-a)/n$ divide the interval (a, b) into equal parts, then

$$\int_a^b F(x)dx=\frac{b-a}{n}\left[\frac{F(a)+F(b)}{2}+\sum_{k=1}^{n-1}F(x)_k\right]-\frac{(b-a)^3}{12n^2}F''(\xi).$$

E. F. Beckenbach (Los Angeles, Calif.).

Prodi, Giovanni. Tracce sulla frontiera delle funzioni di Beppo Levi. Rend. Sem. Mat. Univ. Padova **26** (1956), 36–60.

Poichè posso supporre che atto di cortesia abbia voluto essere il rimettere a me la revisione di questa Nota, mi permetto di incominciare osservando che totalmente ignoro chi sia stato il primo a scegliere il mio nome "Beppo Levi" come strumento di classifica spaziale o funzionale; e se quel primo autore si fosse a me rivolto, lo avrei pure pregato di smettere. Vero è infatti che precisamente poco più di 50 anni fa, nel comporre una Memoria [Rend. Circ. Mat. Palermo **22** (1906), 293–359], ho trovato necessario premettere fin dalle prime linee, e più estesamente nei paragrafi 1 e 3, alcune considerazioni che a quelle iniziali del presente lavoro di G. Prodi si avvicinano, ma la ragione era allora che argomento di attualità era una discussione intorno a una questione variazionale che, sull'esempio di Riemann, precisamente si attribuiva a Dirichlet. Si trattava di un problema di minimo riguardo all'integrale di Dirichlet, del quale in questo lavoro si considera una generalizzazione, supponendo allora arbitrariamente assegnato il valore al contorno della funzione a determinarsi; e poichè è ben certo che la parola funzione, quando non si precisa se razionale, o algebrica, o analitica, o con qualunque altra qualifica che sia del caso, acquista un significato tanto indeterminato que più non esiste problema, non potendo io stabilire a priori la classe delle funzioni a cui riferire il principio di Dirichlet, trovai necessario porre alcune condizioni il meno possibile restrittive che di tali funzioni stabilissero la derivabilità dentro limiti tuttavia adatti allo scopo.

Nel caso presente non saprei dire fino a qual punto possa esistere coincidenza fra le condizioni ammesse qui dall'A. e quelle particolarmente dichiarate nel terzo paragrafo della mia memoria; una generalizzazione ad ogni modo risulta qui nel passare da uno spazio bidimensionale a uno di un numero n indeterminato di coordinate e circoscrivendo il campo funzionale dentro una regione cilindrica la quale però non chiede di essere completamente occupata dalle funzioni in via di considerazione, giungendo quindi a dimostrare quattro teoremi, due dei quali stabiliscono condizioni affinchè sulla sezione media del cilindro si formi traccia delle funzioni considerate, mentre gli altri due stabiliscono proprietà caratteristiche dell'operazione di traccia. *B. Levi.*

Marcus, M.; and Lopes, L. Inequalities for symmetric functions and Hermitian matrices. Canad. J. Math. **9** (1957), 305–312.

For $a_i\geqq 0$, $b_i\geqq 0$ $(1\leqq i\leqq n)$ and $1\leqq r\leqq n$, the inequality

$$E_r^{1/r}(a_1+b_1, \cdots, a_n+b_n)\geqq E_r^{1/r}(a_1, \cdots, a_n)+E_r^{1/r}(b_1, \cdots, b_n)$$

is proved, where E_r denotes the rth elementary symmetric function. If $E_{r-1}(a_1, \cdots, a_n)>0$, $E_{r-1}(b_1, \cdots, b_n)>0$, then

$$\mathscr{E}_r(a_1+b_1, \cdots, a_n+b_n)\geqq\mathscr{E}_r(a_1, \cdots, a_n)+\mathscr{E}_r(b_1, \cdots, b_n),$$

where

$$\mathscr{E}_r(a_1, \cdots, a_n)=\frac{E_r(a_1, \cdots, a_n)}{E_{r-1}(a_1, \cdots, a_n)},\quad E_0(a_1, \cdots, a_n)=1.$$

According to the authors, H. Samelson has observed that the first inequality is a consequence of a concavity result of W. Fenchel [C. R. Acad. Sci. Paris **203** (1936), 764–766] on mixed volumes of convex bodies. However Fenchel's result is not used in the authors' inductive proof. The paper concludes with applications to eigenvalues of the sum of two positive semi-definite Hermitian matrices. In particular, if A, B are two such matrices of order n and if $\{\alpha_i\}$, $\{\beta_i\}$, $\{\lambda_i\}$ are the eigenvalues of A, B, $A+B$ respectively, each arranged in increasing order, then

$$E_r^{1/r}(\lambda_1, \lambda_2, \cdots, \lambda_k)\geqq E_r^{1/r}(\alpha_1, \alpha_2, \cdots, \alpha_k)+E_r^{1/r}(\beta_1, \cdots, \beta_2, \beta_k)$$

holds for $1\leqq r\leqq k\leqq n$. *K. Fan* (Oak Ridge, Tenn.).

Babič, V. M. On theorems of inclusion for a limiting exponent. Vestnik Leningrad. Univ. **11** (1956), no. 19, 186–188. (Russian)

Additional remark (hardly an unexpected one) on the limiting case, treated by Il'in [Dokl. Akad. Nauk SSSR (N.S.) **96** (1954), 905–908; MR **16**, 121], of an inequality due to Sobolev. A simple example is produced, showing that equi-continuity no longer holds in the limiting case.

L. C. Young (Madison, Wis.).

Marcus, S. Fonctions monotones de deux variables. Rev. Math. Pures Appl. **1** (1956), no. 2, 17–36.

The author investigates and compares different notions of monotone functions f of two variables (defined in a square J) which are characterized by one of the following conditions: (a) (L. Tonelli) Each of the differences $f(x+\Delta x, y)-f(x, y)$, $f(x, y+\Delta y)-f(x, y)$ avoids a certain sign (separately) for $\Delta x>0$, $\Delta y>0$. (b) The analogue for $f(x+\Delta x, y+\Delta y)-f(x, y+\Delta y)-f(x+\Delta x, y)+f(x, y)$. (c) (R. C. Young and M. Nicolescu) The differences in (a), (b) avoid the same sign for $\Delta x>0$, $\Delta y>0$; in this case f is called "totally monotone". (d) (H. Lebesgue) In every closed domain, f attains its bounds on the frontier. (e) Each "level" E_t of f (i.e. the set of all points at which f takes on the value t) is connected. (f) The function f is monotone in the sense of A. S. Kronrod [Uspehi Mat. Nauk (N.S.) **5** (1950), no. 1(35), 24–134; MR **11**, 648]. Many relations between these notions of monotone functions $f(x, y)$ are established by the author. E.g., he proves the following theorems: The functions of (e) and (f) coincide. The functions of (d) are characterized by the property that each component of E_t (see (e)) which separates J meets the boundary of J. Every function of (a) belongs to the types (d) and (e). On the other hand it

is shown that the functions of (b) are not comparable with those of (a), (d), (e), (f), and also that the functions of (d) and (f) are not comparable. *A. Rosenthal.*

Volpato, Mario. Sulle condizioni sufficienti per la continuità (di ordine n) di un funzionale di ordine $n+1$. Rend. Sem. Mat. Univ. Padova 26 (1956), 1–9.

Generalizing results of S. Cinquini, M. Picone, S. Faedo, and G. Darbo, the author proves the following theorem: Let R be the $(n+2)$-dimensional interval $a \leqq x \leqq b$; $y_1^{(i)} \leqq y^{(i)} \leqq y_2^{(i)}$ ($i=0, 1, \cdots, n$; $y^{(0)}=y$) and let $\mathfrak{K}$ be the class of functions $y(x)$ with absolutely continuous nth derivative in (a, b) such that $y_1^{(i)} \leqq y^{(i)}(x) \leqq y_2^{(i)}$ ($i=0, 1, \cdots, n$; $y^{(0)}=y$); let the real function $Q(x, y, y', \cdots, y^{(n-1)}, y^{(n)})$ be defined and bounded in R and there measurable with respect to $y^{(n)}$ and continuous with respect to $(x, y, y', \cdots, y^{(n-1)})$ for almost all values of $y^{(n)}$ in $[y_1^{(n)}, y_2^{(n)}]$; let there exist $n+1$ functions $\mathfrak{L}(u)$, $L_0(u)$, $L_1(u)$, $\cdots$, $L_{n-1}(u)$, non-negative and summable in $y_1^{(n)} \leqq u \leqq y_2^{(n)}$, such that for almost all values of u in $[y_1^{(n)}, y_2^{(n)}]$ one has

$$|Q(x, y, y', \cdots, y^{(n-1)}, u) - Q(\bar{x}, \bar{y}, \bar{y}', \cdots, \bar{y}^{(n-1)}, u)| \leqq \mathfrak{L}(u)|x-\bar{x}| + L_0(u)|y-\bar{y}| + L_1(u)|y'-\bar{y}'| + \cdots + L_{n-1}(u)|y^{(n-1)} - \bar{y}^{(n-1)}|,$$

where $(x, y, y', \cdots, y^{(n-1)}, u)$, $(\bar{x}, \bar{y}, \bar{y}', \cdots, \bar{y}^{(n-1)}, u)$ are points of R. Then the integral

$$\mathfrak{J}[y(x)] = \int_a^b Q[x, y(x), y'(x), \cdots, y^{(n)}] y^{(n+1)}(x) dx$$

is a continuous functional (of order n) in the class $\mathfrak{K}$.
A. Rosenthal (Lafayette, Ind.).

Gagliardo, Emilio. Un criterio di eguale continuità per funzioni di due variabili. Rend. Sem. Mat. Univ. Padova 26 (1956), 148–167.

The following theorems are proved: (1) Let D be the plane domain $\{y_0 \leqq y \leqq y_1,\ \varphi(y) \leqq x \leqq \psi(y)\}$ $(\varphi(y) < \psi(y))$, where the functions $\varphi(y)$, $\psi(y)$ are Hölderian of order $\lambda > \frac{1}{2}$ for $y_0 \leqq y \leqq y_1$; let $\{u_n(x, y)\}$ be a uniformly bounded sequence of continuous functions with derivatives $\partial u_n/\partial x$, $\partial u_n/\partial y$, $\partial^2 u_n/\partial x^2$ in D; let there exist two positive constants α, A such that, for every n,

$$(*) \quad \iint_D \left[\left| \frac{\partial u_n}{\partial y} \right|^{1+\alpha} + \left| \frac{\partial^2 u_n}{\partial x^2} \right|^{1+1/\alpha} \right] dxdy \leqq A \quad (0 \leqq \alpha \leqq 1).$$

Then the functions $u_n(x, y)$ are equi-Hölderian in D, and hence the sequence $\{u_n(x, y)\}$ is compact in D with respect to uniform convergence. (2) (Without the assumption of uniform boundedness): Let the domain D be as in (1), but with $\varphi(y)$, $\psi(y)$ being Hölderian of order $\lambda > \frac{1}{2}$; let $\{u_n(x, y)\}$ be a sequence of continuous functions with derivatives $\partial u_n/\partial x$, $\partial u_n/\partial y$, $\partial^2 u_n/\partial x^2$ in D; let there exist two positive constants α, A such that, for every n,

$$(**) \quad \iint_D [\,] \left| \frac{\partial u_n}{\partial x} \right|^{\max[1+\alpha,\, 1+1/\alpha]} + \left| \frac{\partial u_n}{\partial y} \right|^{1+\alpha} + \left| \frac{\partial^2 u_n}{\partial x^2} \right|^{1+1/\alpha} \Bigg] dxdy \leqq A.$$

Then the functions $u_n(x, y)$ are equi-continuous in D. (3) In (2) one can replace (**) by (*) together with the assumption that

$$\iint_D |u_n(x, y)|^{1+1/\alpha} dxdy \leqq M.$$

Finally, the author gives several examples to show that some of his assumptions are essential. *A. Rosenthal.*

Hummel, J. A. Counterexamples to the Poincaré inequality. Proc. Amer. Math. Soc. 8 (1957), 207–210.

Sofern das ebene Gebiet D genügend regulär ist, gilt die Poincarésche Ungleichung

$$\iint_D \phi^2 dxdy \leqq K(D) \iint_D (\phi_x^2 + \phi_y^2) dxdy$$

unter der Normierungsbedingung $\iint_D \phi dxdy = 0$ für alle stetigen ϕ mit endlichem Dirichletintegral. Dass sie nicht für beliebige Jordangebiete gelten kann zeigt ein Beispiel in Courant und Hilbert, Methoden der mathematischen Physik [Bd. 2, Springer, Berlin, 1937, p. 521]. Die dort konstruierte Funktion ϕ ist sehr irregulär. Verf. gibt nun ein Gebiet D an, worin sogar eine harmonische Funktion ϕ existiert, für welche die Poincarésche Ungleichung nicht gelten kann: es ist $\iint_D \phi dxdy = 0$, $\iint_D (\phi_x^2 + \phi_y^2) dxdy < \infty$ und $\iint_D \phi^2 dxdy = \infty$. D ist allerdings nicht mehr ein Jordangebiet: D entsteht aus dem Einheitskreis $|z| < 1$, indem man ihn von einem innern Punkt aus längs einer Spirale aufschneidet, die sich asymptotisch der Peripherie nähert. Für dasselbe Gebiet konstruiert Verf. eine holomorphe Funktion f mit $f(z_0) = 0$, $z_0 \in D$, $\iint_D |f'(z)|^2 dxdy < \infty$ und $\iint_D |f(z)|^2 dxdy = \infty$.
A. Pfluger (Zürich).

★ **Болтянский, В. Г. [Boltyanskiĭ, V. G.] Что такое дифференцирование? [What is differentiation?]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 63 pp. 90 kopeks.

Attempt to explain in a form accessible to students of higher grades of secondary schools certain concepts of higher mathematics such as derivative, differential equation, number e, natural logarithm; it is brought out that such concepts are reflections of real processes that take place in nature.

Intuitive aspects are stressed sometimes at the expense of rigor.

★ **Boltjanski, W. G. Differentialrechnung einmal anders.** VEB Deutscher Verlag der Wissenschaften, Berlin, 1956. v+60 pp.

A translation by E. G. Lierke and R. Lindner of the book reviewed above.

Ostrowski, Alexander. Über die Differenzierbarkeit von impliziten Funktionen. Verh. Naturf. Ges. Basel 67 (1956), 141–148.

Let $F(u, x) = 0$, $u = u(x)$. If F_x', F_u' are continuous then (a) $du/dx = -F_x'/F_u'$ at each point x_0 at which $F_u' \neq 0$. (The notation $u(x_0) = u^0$, $F_x'(u_0, x_0) = F_x^0$, $F_u'(u_0, x_0) = F_u^0$, $F_{uu}''(u_0, x_0) = F_{uu}^0$ is used.) Thus when $u'(x)$ becomes infinite at a common point of continuity of u, F_u', F_x' then this manifests itself in the vanishing of F_u^0. In general F_x^0 does not vanish so that the infinity of u' comes about by means of (a). This does not occur if $u(x)$ satisfies at x_0 a Lipschitz condition of order $\alpha > \frac{1}{2}$ and F_{uu}^0 exists. In this case F_x^0 also vanishes and (a) becomes

$$\text{(b)} \qquad \left[\frac{d}{dx} (F_u^0 u) \right]_{x=x_0} = -F_x^0.$$

The requirement that F_{uu}^0 exist can be weakened to

requiring that $F'(u, x_0)$ satisfies a Lipschitz condition of order $>1/\alpha-1$. It is shown that the Lipschitz condition for u and F_u' cannot be improved but that the formula can be improved if a more general type of Lipschitz condition is introduced.

The relation (b) can be generalized to functions $u_1(x), \cdots, u_n(x)$ and $F(u_1, \cdots, u_n, x)=0$. If F, F_{u_ν}' $(\nu=1, \cdots, n)$ and $F'(x)$ are continuous for $x=x_0$ and for the corresponding values u_ν^0 and if $F_{u_\nu u}''$ exists at (u_ν^0, x_0) then if all the $u_\nu(x)$ satisfy a Lipschitz condition of order $>\frac{1}{2}$ at x_0 there exists a linear combination of the $u_\nu(x)$ which is differentiable at x_0 with respect to x. If F_x', F_{u_ν}' are at (x_0, u_ν^0) denoted by $F_x^0, F_{u_\nu}^0$ then

(c) $$\left[\frac{d}{dx}\sum_{\nu=1}^{n} F_{u_\nu}{}^0 u_\nu\right]_{x=x_0}=-F_x^0,$$

and again the existence of F_{uu_ν}'' can be replaced by the condition that F_{u_ν}' satisfy a Lipschitz condition of order $>1/\alpha-1$. A function f at (u_u^0) belongs to the class L_χ, $\chi>0$, with respect to v_u, if

(d) $$|f(v_u)-f(v_u^0)|<C(\sum_u|v_u-v_u^0|)^\chi$$

for all points v_u in a given region of v_u^0, C depending only on f. Let $\varepsilon>0$ be arbitrary. If there exists δ_ε such that when $\sum_u |v_u-v_u^0|<\delta_\varepsilon$, the number ε can replace C in (d), then f belongs to class l_χ. With this notation there is obtained the following theorem. Let $u_\nu(x_1, \cdots, x_m)$ be continuous at $P_0(x_1^0, \cdots, x_m^0)$ and satisfy $F(u_\nu; x_\mu)=0$ in a neighborhood of P_0. Denote the n-dimensional point (u_ν^0) by Q_0, the $m+n$ dimensional point $(u_\nu^0; x_\mu^0)$ by R_0. Let the derivatives $F_{u_\nu}'(u_\nu, x_\mu)$ exist and be continuous in a neighborhood of R_0. Also let one of the two following conditions be fulfilled. For α with $0<\alpha<1$ all the u_ν belong to the class l_α with respect to the x_ν at P_0, while the derivatives $F_{u_\nu}'(u_1, \cdots, u_n; x_1^0, \cdots x_m^0)$ belong to the class $1/\alpha-1$ with respect to $u_1, \cdots, u_n$ at Q_0. For α with $0<\alpha\leq 1$ all u_ν belongs to α with respect to x_ν at P_0, while the derivatives

$$F_{u_\nu}'(u_1, \cdots, u_n; x_1^0, \cdots, x_n^0)$$

belong to the class $l_{1/\alpha-1}$ with respect to $u_1, \cdots, u_n$ at Q_0. If one denotes the derivatives F_{x_u}', F_{u_ν}' at R_0 by $F_{x_u}{}^0$, $F_{u_\nu}{}^0$ respectively then $\sum F_{u_\nu}{}^0 u_\nu$ possesses a differential in the Stolz senses at P_0 and

$$d\sum_{\nu=1}^{n} F_{u_\nu}{}^0 u_\nu=-\sum_{u=1}^{n} F_{x_\mu}{}^0 dx_\mu.$$

R. L. Jeffery (Kingston, Ont.).

Geronimus, Ya. L. On differential properties of certain functions represented by singular integrals. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 775–782. (Russian)

The author considers differential properties of functions $v(\sigma)$ representable by singular integrals

(*) $$\int_0^l K(s-\sigma)d\mu(s) \quad (0<\sigma<l),$$

where $K(s)$ is an odd function having a singularity at $s=0$ only, satisfying $\int_0^l |K(x)|dx=\infty$, and the integral (*) is taken in the principal value sense. The main result is as follows. If on a certain segment (s_1-h, s_2+h) interior to $(0, l)$ the function $\mu(s)$ has a continuous derivative of order k whose modulus of continuity ω_k satisfies the condition

$$\int_0^a \omega_k(x)|K(x)|dx<\infty,$$

and if $K(x)$ has a continuous kth derivative outside each neighborhood of $x=0$, then $v(\sigma)$ has a bounded $(k-1)$th derivative on (s_1, s_2). The most significant applications are to Cauchy-Stieltjes integrals

$$v(\sigma)=\int_L \frac{d\mu(s)}{s-\sigma}.$$

A. Zygmund (Chicago, Ill.).

Corominas Vigneaux, Ernesto. On the characterization of derivatives defined as limits of divided differences. Rev. Acad. Ci. Madrid **49** (1955), 115–212. (Spanish)

Starting with the observation that notions of Newton and Leibniz led to equivalent definitions of the derivative, the author notes that there are other notions that are related to these but that do not lead to equivalent definitions. Thus, as is well known, for a function $f(x)$ that is continuous at the origin the existence of

$$\lim[f(h)-f(-h)]/(2h)$$

as $h\to 0$ does not imply the existence of $f'(0)$, nor does the existence of $f'(0)$ imply the existence of

$$\lim [f(x_2)-f(x_1)]/(x_2-x_1)$$

as $x_1\to 0$, $x_2\to 0$ with $x_1\neq x_2$.

The author now discusses at length relations between derivatives, limits of difference quotients, and differential coefficients of the first and higher orders, reviewing known results and adding some of his own. For example, extending a generalization by Á. Császár [Ann. Sci. École Norm. Sup. (3) **64** (1947), 275–284; MR **10**, 109] of a theorem of Stieltjes, the author shows that if $f(x)$ has an nth derivative $f^{(n)}(x_\omega)$ at x_ω, then the nth divided difference $\Delta^n[x_1, \cdots, x_{n+1}; f(x)]\equiv\Delta^n[x_j; f(x)]$ satisfies

$$\lim \Delta^n[x_j; f(x)]=(n!)^{-1}f^{(n)}(x_\omega) \text{ as } x_j\to x_\omega \ (j=1, \cdots, n+1)$$

subject to the restriction that $d(x_\omega, x_j)/d(x_j)\leq K$, where K is a constant ≥ 1 and $d(x_\omega, x_j)$ and $d(x_j)$ denote the diameters of the sets indicated.

E. F. Beckenbach (Los Angeles, Calif.).

See also: Cugiani, p. 875; Vaida, p. 890; Timan, p. 890; Gilbert, p. 894; Boas, p. 906; Ghermănescu, p. 944.

Measure, Integration

Lombardi, Federico. Sugli insiemi analitici dell'S_r euclideo. Boll. Un. Mat. Ital. (3) **11** (1956), 578–581.

The statement of the author is contained in the more general theorem 7.1.211 of H. Hahn and A. Rosenthal, "Set functions" [Univ. of New Mexico Press, 1948; MR **9**, 504], which gives a condition for the φ-measurability of every analytic set of a metric space and which is an immediate consequence of the theorems 6.3.5 (referred to by the author) and 6.3.51. The only contribution of the author is a new proof of a lemma (corresponding to theorem 6.3.32 of Hahn-Rosenthal [loc. cit.]).

A. Rosenthal (Lafayette, Ind.).

Cafiero, Federico. Teoremi di prolungamento per le misure relative in particolari reticoli d'insiemi. Ricerche Mat. **5** (1956), 273–312.

The author modifies some of his previous definitions [Ann. Mat. Pura Appl. (4) **40** (1955), 269–283; MR **17**, 719] and deals with appropriate variants and extensions of his earlier results. *L. C. Young* (Madison, Wis.).

Scorza Dragoni, Giuseppe. Sulla derivazione degli integrali indefiniti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 711–714.

The terminology used in this paper agrees with that in Saks, "Theory of the integral" [2nd ed., Warszawa, 1937, see, in particular, p. 106]. The paper contains several useful results concerning the derivation of definite integrals. We quote the following theorem in the way of illustration. In Euclidean n-space let f and F be real-valued Lebesgue-measurable functions such that $|f|\leqq F$ at every point and F is Lebesgue-summable on every bounded, Lebesgue-measurable set E. For such sets E put $\varphi(E)=\int_E f$, $\phi(E)=\int_E F$, the integrals being Lebesgue integrals. Then the ordinary derivative of φ exists and is equal to f at every point x, where (i) F is the ordinary derivative of ϕ and (ii) f and F are both approximately continuous. The proof is made by reduction to the classical theorem on the derivation of the indefinite integral of a bounded function. Further results, of a similar character, are concerned with general and strong derivatives.

T. Radó (Columbus, Ohio).

Mařík, Jan. Transformation of m-dimensional Lebesgue integrals. Czechoslovak Math. J. **6(81)** (1956), 212–216. (Russian summary)

Let J be the Jacobian of a continuously differentiable mapping T, into Euclidean m-space, of an open subset G of this space, and let $N(x)$ be the set $T^{-1}(x)$. Suppose further that f is a real function such that $|J|f$ has, on G, the integral I (finite or infinite). Then I is the integral on $T(G)$ of the function $g(x)=\sum_{t\epsilon N(x)} f(t)$, and in particular this function exists almost everywhere in $T(G)$.

L. C. Young (Madison, Wis.).

Henstock, R. The summation by convergence factors of Laplace-Stieltjes integrals outside their half plane of convergence. Math. Z. **67** (1957), 10–31.

In this paper, the summability of a Laplace-Stieltjes integral $\int_0^\infty e^{-zw}ds(w)$ by convergence factors $a(x,w)$ in regions outside the half-plane of convergence is studied.

Before stating the principal results, some explanations of terms employed in the enunciations must be given. A Baire (Borel measurable) function is any function which can be obtained from continuous functions by using repeated limits. It will be supposed that $a(x,w)$ is a finite Baire function satisfying

$$(1) \qquad \liminf_{u\to w} a(x,u)\leqq a(x,w)\leqq \limsup_{u\to w} a(x,u),$$

when w and u are in $w\geqq 0$. The author, in a previous paper [J. London Math. Soc. **30** (1955), 273–286; MR **17**, 359], defined the "integral by parts" to be

$$(2) \quad (\mathrm{IP})\int_\alpha^\beta f(w)ds(w)=f(\beta)s(\beta)-f(\alpha)s(\alpha)-(\mathrm{PS})\int_\alpha^\beta s(w)df(w)$$

where (PS) denotes Ward's [Math. Z. **41** (1936), 578–604] Perron-Stieltjes integral. The Laplace-Stieltjes integrals are taken to be of the form

$$(3) \qquad F(s(\cdot);z)\equiv F(z)\equiv \lim_{w\to\infty} F(z,w),$$

where

$$(4) \quad F(z,w)=(\mathrm{IP})\int_0^w e^{-zu}ds(u)= e^{-zw}s(w)-s(0)+(\mathrm{L})\int_0^w ze^{-zu}s(u)du,$$

$s(u)$ being a Baire function bounded in each finite interval of $u\geqq 0$, and (L) denoting the Lebesgue integral.

An H-set is defined to be a set Z of complex numbers such that (i) Z is closed; (ii) if v lies in Z, then so do all points z with $\Im(z)=\Im(v)$, $\Re(z)\leqq\Re(v)$; (iii) the zero 0 lies in Z; (iv) no point of $\Re(z)>0$ lies in Z; (v) the section of Z by each line $\Re(z)=$const is bounded. $R(\sigma)$ will denote the half-plane $\Re(z)>\sigma$.

For summation of the Laplace-Stieltjes integrals given by (3) and (4), the author considers

$$(5) \qquad B(F;z,x)\equiv(\mathrm{IP})\int_0^\infty a(x,w)d_wF(z,w),$$

where $a(x,w)$ satisfies (1); he proves (in Lemma 1) that if $a(x,w)$ is bounded in each finite interval of $w\geqq 0$, then

$$B(F;z,x)=(\mathrm{IP})\int_0^\infty a(x,w)e^{-zw}ds(w).$$

Let Z be an H-set; taking a fixed real number p, the function $a(x,w)$ is said to be efficient $(Z;p)$ for an $F(z)$ satisfying (3) and (4) if $B(F;z,x)$ exists for each $x\geqq 0$ and each z in $R(\sigma-p)$, and, as $x\to\infty$, $B(F;z,w)$ is uniformly convergent to $F(z)$ in each compact set contained in $H(F;Z)\cap R(\sigma-p)$, whenever $F(z)$ converges in $R(\sigma)$, where $H(F;Z)\equiv C\bigcup_\beta\{Z+(\alpha+i\beta)\}$. Finally, $\mathrm{V}[f;W]$ denotes the weak variation of $f(w)$ over a set W in $w\geqq 0$ [See S. Saks, Theory of the Integral, 2nd. ed., Warsaw, 1937, p. 221, § 4].

Necessary and sufficient conditions on $a(x,w)$ are then found in order that it should be efficient $(Z;p)$ for all $F(z)$ satisfying (3) and (4), or for a special subclass of $F(z)$. The chief result in this direction is Theorem 1, as follows: If $a(x,w)$ is a finite Baire function satisfying (1), then in order that the $B(F;z,w)$ of (5) should exist for a fixed $x\geqq 0$, for z in $R(\sigma-p)$, and for each $F(z)$ satisfying (3), (4), convergent in $R(\sigma)$,and with $s(w)$ a Baire function bounded in each finite interval of $w\geqq 0$, it is necessary and sufficient that

$$M_1(q,x)\equiv \mathrm{V}[e^{qw}a(x,w);(0,\infty)]<\infty \text{ for each } q<p.$$

In Theorem 2, the similar problem for a subclass S_r of the functions $F(z)$ is solved; for $r\geqq 0$, $F(z)$ is said to belong to the class S_r if $F(z)$ satisfies (4), with $s(w)$ a Baire function satisfying

$$s(w)=O(1),\ (\mathrm{L})\int_w^{w+1}|s(u)|du=O(e^{-rw}) \text{ as } w\to\infty.$$

In Theorem 3, the general class of functions $F(z)$ is dealt with, the $a(x,w)$ then satisfying an extra condition (p. 22, (55)]. Finally, in Theorem 4, the preceding theory is related to Okada's theory [Math. Z. **23** (1925), 62–71] and the general theory of summability of Taylor series in star domains, principal or partial [Cooke, Infinite matrices and sequence spaces, Macmillan, London, 1950, Ch. 8; MR **12**, 694].

R. G. Cooke (London).

Henstock, Ralph. On Ward's Perron-Stieltjes integral. Canad. J. Math. **9** (1957), 96–109.

Let $P(f,g;a,b)=\int_a^b f(u)dg(u)$ denote the Perron-Stieltjes integral, where f and g are finite functions defined in $[a,b]$, $j(u)=g(u)-\frac{1}{2}\{g(u^+)-g(u^-)\}$, $E_\varepsilon=\{u;|g(u)|\geqq\varepsilon>0\}$. a) If $P(f,g;a,u)=[fg]_a^u$, for all $a\leqq u\leqq b$, then f is VBG and continuous on $\bar{E}_\varepsilon$, and the measure of $\bar{E}_\varepsilon=0$. b) $P(f,g;a,b)$ exists for all bounded Baire functions f in $[a,b]$, if and only if: (i) $g(u^-)$ and $g(u^+)$ are of bounded variation; (ii) $\{u;j(u)\neq 0\}=\{u_n\}\cup\{d_n\}$ $(n=1,2,\cdots)$; (iii) $\sum_n|j(u_n)|<\infty$; (iv) surrounding each d_n there is an

interval $I(d_n)\equiv(d_n', d_n'')$ such that each point of $[a, b]$ belongs to an at most finite number of the $I(d_n)$, and there is an increasing bounded function χ such that $\chi(d_n''+)-\chi(d)_n\geqq|j(d_n)|, \chi(d_n)-\chi(d_n'-)\geqq|j(d_n)|$. c) The author considers the case $j(u^-)=0$ $(a<u\leqq b)$, $j(u^+)=0$ $(a\leqq u<b)$, and studies the sets T_1 and W, where $T_1=$union of the interiors of all segments J such that $P(f, j; J)$ exists for all bounded Baire functions f, and $W=$the set of points of infinite variation of j. CT_1 is empty if W is countable, and if and only if the set $\{d_n\}$ is scattered, but no structural property of W can be both necessary and sufficient for CT_1 to be empty. *M. Cotlar* (Buenos Aires).

Haupt, Otto. Über die Entwicklung des Integralbegriffes seit Riemann. Schr. Forschungsinst. Math. **1** (1957), 303–317.

This is an expository article, largely an abstract of material contained in Haupt, Aumann, and Pauc, Differential und Integralrechnung, v. 3, 2nd ed. [Springer, Berlin, 1955; MR **17**, 1066]. The first section considers Riemann integral in n-dimensional space contrasting the two modes of convergence of approximating sum, Jordan content, Lebesgue measure, Riemann integral of functions on bi-compact spaces, general integral on the basis of successive subdivision of the measure lattice, and Carathéodory's "algebraic" approach to integration. The second section considers the integral as linear functional starting from the Lebesgue postulates, includes the Stone approach to extensions via a semi-norm and discusses the Bourbaki theory of Radon functionals as well as the procedure of A. D. Alexandroff. Emphasized are the role of topology relative to measure in the spaces on which the integrated functions are defined, and the various modes of extension. *T. H. Hildebrandt* (Ann Arbor, Mich.).

Aumann, Georg. Über Typen von Zerlegungsausrichtungen in der allgemeinen Integrationstheorie. Jber. Deutsch. Math. Verein. **59** (1956), Abt. 1, 79-86.

Notations and Definitions. $\mathfrak{A}$: Boolean algebra with unit A. $\mathfrak{J}$: set of non-empty elements of $\mathfrak{A}$, called "cells", such that any nonempty intersection of two of them is a union of disjoint cells. I, I', I_ν: cells. $\mathfrak{J}$- (or cell) partition of $\mathfrak{A}$: finite set of disjoint cells whose union is $\mathfrak{A}$. $\mathfrak{z}, \mathfrak{z}', \mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3$: $\mathfrak{J}$-partitions of $\mathfrak{A}$. $\mathfrak{Z}$: set of all $\mathfrak{J}$-partitions of $\mathfrak{A}$. $(\mathfrak{Z}, \geqslant)$: directed set with carrier $\mathfrak{Z}$. $(\mathfrak{z}_1\geqslant_U\mathfrak{z}_2)$: any $\mathfrak{z}_1$-cell is included in some $\mathfrak{z}_2$-cell. $\mathfrak{H}_U(\mathfrak{z})$: set of the subcells of the $\mathfrak{z}$-cells. $N(I)$ (cell norm): one-valued real function defined on $\mathfrak{J}$ such that $(I\subset I')\Rightarrow(N(I)\leqq N(I'))$. $N(\mathfrak{z})$: $\max(N(I_\nu); I_\nu\in\mathfrak{z})$. $(\mathfrak{z}_1\geqslant_N\mathfrak{z}_2)$: $(N(\mathfrak{z}_1)\leqq N(\mathfrak{z}_2))$. Cell function Q: many-valued real function defined on $\mathfrak{J}$. $\underline{S}(\geqslant)$, $\overline{S}(\geqslant)$: lower, upper Burkill $\geqslant$-integral of Q on $\mathfrak{A}$. Two directions $\geqslant_1$ and $\geqslant_2$ are "equivalent" if corresponding to any $\mathfrak{z}\in\mathfrak{Z}$ there exist $\mathfrak{z}_1$ and $\mathfrak{z}_2$ in $\mathfrak{Z}$ such that $(\mathfrak{z}'\geqslant_2\mathfrak{z}_1)\Rightarrow(\mathfrak{z}'\geqslant_1\mathfrak{z})$, $(\mathfrak{z}'\geqslant_1\mathfrak{z}_2)\Rightarrow(\mathfrak{z}'\geqslant_2\mathfrak{z})$. A direction $\geqslant$ is cell "determined" if for any two cell partitions $\mathfrak{z}=\{I_1, \cdots, I_n\}$, $\mathfrak{z}'$ the existence for each $\nu=1, \cdots, n$ of a partition $\mathfrak{z}_\nu$ with $I_\nu\in\mathfrak{z}_\nu$ and $\mathfrak{z}_\nu\geqslant\mathfrak{z}'$ implies $\mathfrak{z}\geqslant\mathfrak{z}'$. Th. 1. Two cell determined directions $\geqslant_1$ and $\geqslant_2$ are equivalent if and only if $\overline{S}(\geqslant_1)=\overline{S}(\geqslant_2)$ for any cell function Q. Th. 2. A direction $\geqslant$ is cell determined if and only if there exists a mapping $\mathfrak{K}$ of $\mathfrak{Z}$ into the set of subsets of $\mathfrak{J}$ possessing the following properties: (1') $\mathfrak{z}\subset\mathfrak{K}(\mathfrak{z})$, (2') $((\mathfrak{z}_1\subset\mathfrak{K}(\mathfrak{z}_2))\&(\mathfrak{z}_2\subset\mathfrak{K}(\mathfrak{z}_3))\Rightarrow(\mathfrak{z}_1\subset\mathfrak{K}(\mathfrak{z}_3)))$, (3') for any $\mathfrak{z}_1, \mathfrak{z}_2$, the intersection $\mathfrak{K}(\mathfrak{z}_1)\frown\mathfrak{K}(\mathfrak{z}_2)$ includes an $\mathfrak{J}$-partition, (4') $(\mathfrak{z}'\geqslant\mathfrak{z})\Leftrightarrow(\mathfrak{z}'\subset\mathfrak{K}(\mathfrak{z}))$. Def. A direction $\geqslant$ is "normal" if there exists a mapping $\mathfrak{K}$ fulfilling the requirements of Th. 2. and in addition (1'') $\mathfrak{K}(\mathfrak{z})\supset\mathfrak{H}_U(\mathfrak{z})$, (2'') $(\mathfrak{z}'\subset\mathfrak{K}(\mathfrak{z})\Rightarrow(\mathfrak{K}(\mathfrak{z}')\subset\mathfrak{K}(\mathfrak{z}))$. Examples. $\geqslant_U$ and $\geqslant_N$ are normal. Concrete case. $A=[0, 1)$. $\mathfrak{J}$: set of intervals $[a, b)$ with $0\leqq a<b\leqq 1$. $N(I)$: length of I. A direction $\geqslant$ is called "infinitair" if corresponding to any $\delta>0$ there exists a cell partition $\mathfrak{z}$ such that any cell of any partition $\mathfrak{z}'\geqslant\mathfrak{z}$ has a norm $<\delta$. A simple construction is given for all directions which are both normal and infinitair, provided with an illuminating illustration in the plane. *C. Pauc* (Nantes).

★**Bourbaki, N. Eléments de mathématique. XXI. Première partie: Les structures fondamentales de l'analyse. Livre VI: Intégration. Chapitre V: Intégration des mesures.** Actualités Sci. Ind., no. 1244. Hermann & Cie, Paris, 1956. ii+131 pp.

In a locally compact space X a positive integral μ is defined by extension from $K(X)$, the space of continuous functions with compact support in X, first to non-negative lower semicontinuous functions by taking upper bounds of μ, then to arbitrary non-negative functions by taking lower bounds. The closure of $K(X)$ relative to this norm μ^* defines the class of μ-integrable functions. In the present chapter a sharper norm $\bar{\mu}^*(f)=\sup\mu^*(f\varphi_K)$ is introduced, where φ_K is the characteristic function of K and K runs over all compact subsets of X. Functions integrable relative to this norm are called essentially μ-integrable, and the null functions are called locally negligible. The effect of this new extension is only to enlarge the class of null functions, since any essentially μ-integrable function is equivalent to a μ-integrable function. This integral is the basis for most of the developments in this chapter. For instance, let μ be a positive measure on a locally compact space T. A family λ_t $(t\in T)$ of positive measures on X is called μ-adequate if $\lambda_t(f)$ is essentially μ-integrable for each $f\in K(X)$, and if the mapping $t\to\lambda_t$ is vaguely μ-measurable. Then $\nu=\int\lambda_t d\mu(t)$ is defined, and for every ν-integrable f (real or Banach space valued) the function $t\to\int f(x)d\lambda_t(x)$ is defined for all t except a locally negligible set, is essentially μ-integrable, and $\int f(x)d\nu(x)=\int d\mu(t)\int f(x)d\lambda_t(x)$. The Lebesgue-Fubini theorem is obtained from this by specialization. Let π map T into X, and let g be a nonnegative function on T. Let ε_x denote the unit measure concentrated at the point $x\in X$. The pair (π, g) is called μ-adapted if π and g are μ-measurable and the mapping $t\to f(\pi(t))g(t)$ is essentially μ-integrable for every $f\in K(X)$. Then $\nu=\int g(t)\varepsilon_{\pi(t)}d\mu(t)$ is defined. A function f is essentially ν-integrable if and only if $(f\circ\pi)g$ is essentially μ-integrable, and then

$$\int f(x)d\nu(x)=\int f(\pi(t))g(t)d\mu(t).$$

This and related results form the basis for a study of mappings of one measure space in another, and of induced measures. Another section treats measures defined by a density function, and the Lebesgue-Nikodym theorem. A large number of special results are formulated as exercises, which cover a total of more than 31 pages. The fascicule concludes with a 12 page historical note, outlining the development of the notion of integral from Euler to the present day. *J. C. Oxtoby.*

Pagni, Mauro. Sulla derivazione negli insiemi astratti delle funzioni a variazione limitata. Rend. Sem. Mat. Univ. Padova **26** (1956), 61–69.

The present note is a continuation of same Rend. **25** (1956), 279–302 [MR **18**, 118] whose notations and definitions are taken over. F: real function defined on the semi-ring $\{I\}$ and of bounded variation. $G(I)$, $\overline{G}(I)$: lower upper Burkill (partition) integral of F over I; both are

additive and of bounded variation. The lower μ-derivate $\underline{F}_B'$ and the upper μ-derivate $\overline{F}_B'$ are defined as the μ-derivatives of $\underline{G}$ and $\overline{G}$ respectively; if they are μ-equivalent, F is said to be μ-derivable and $F_B'=\underline{F}_B'=\overline{F}_B'$ is called the μ-derivative of F. F is termed μ-singular if $\underline{F}_B'=\overline{F}_B'=0$. Some results of Burkill's theory of interval functions are obtained in a very simple manner using the definitions above; here are the main ones: Theorem IV. If $F(I)$ is μ-continuous, $\underline{G}(I)=\int_I \underline{F}_B'\cdot d\mu$, $\overline{G}(I)=\int_I \overline{F}_B'\cdot d\mu$. Theorem V. If $F(I)$ is μ-derivable, $F(I)=F^*(I)+\int_I F_B'\cdot d\mu$ where F^* is μ-singular; this decomposition is unique. In the second part of the paper $\{I\}$ denotes the family of the mixed intervals $[a;b)$ of the euclidean space S_r, μ the Lebesgue measure; the extreme regular derivates $\underline{D}_F(x)$ and $\overline{D}_F(x)$ are compared with $\underline{F}_B'$ and $\overline{F}_B'$: Theorem VIII. Almost everywhere $\underline{F}_B'(x)\leqq\underline{D}_F(x)\leqq\overline{D}_F(x)\leqq\overline{F}_B'(x)$. A counter-example shows that $\underline{D}_F(x)$ may be $=\overline{D}_F(x)$ almost everywhere without F being μ-derivable.

C. Pauc (Nantes).

Cesari, Lamberto. Properties of contours. Rend. Mat. e Appl. (5) **15** (1956), 341–365 (1957).

The paper is one of a series, written in part by, or jointly with, R. E. Fullerton [cf., e.g., Cesari, Riv. Mat. Univ. Parma **4** (1953), 173-194; MR **15**, 611; Fullerton, ibid. **4** (1953), 207–212; MR **15**, 612], devoted to reformulating the theory of prime ends and applying it to Surface Area, as in Chapter VI of the author's book "Surface area" [Princeton, 1956; MR **17**, 596]. One feature of Cesari's reformulation of this approach is that it no longer requires the passing use of an area different from Lebesgue's, as in some of the reviewer's lemmas [Fund. Math. **35** (1948), 275–302; MR **10**, 520]. In this paper, the author introduces the notion of m-continuum and uses it to study prime ends and their wings. He then goes on to define, for mappings into E_3, his generalized length and his smoothing process in a more general setting than that of Chapter VI of his book. The smoothing process has the advantage of leading to continua which can be approached from both sides, and a number of applications are promised. *L. C. Young.*

Civin, Paul. Correction to "Some ergodic theorems involving two operators". Pacific J. Math. **6** (1956), 795.

Correction to same J. **5** (1955), 869–876 [MR **17**, 833].

J. Schwartz (New York, N.Y.).

See also: Linnik, p. 874; Marcus, p. 877; Ciorănescu, p. 876; Cotlar, p. 893; Ul'yanov, p. 892; Blackwell, p. 940; Harris, p. 941.

Functions of Complex Variables

Rinehart, R. F. Extension of the derivative concept for functions of matrices. Proc. Amer. Math. Soc. **8** (1957), 329–335.

By a function $f(Z)$ of a matrix Z is meant a mapping of a subset into the set of all square matrices of order n over the real (complex) field. Previous attempts at definitions of differentiability and derivative for such functions have not been entirely satisfactory. For instance H. Richter [Math. Ann. **122** (1950), 16–34; MR **12**, 235] showed that if $f(z)$ is an analytic function of a complex variable, A is a matrix and t is a parameter, then

$$f(A+tD)-f(A)=tf'(A)D+O(t^2)$$

provided the matrix D commutes with A. A definition of derivative, however, based on this result restricts the matrix increment to one which commutes with the argument A. The author of the present paper defines $f(Z)$ to be differentiable at $Z=A$ if for all H in a sufficiently small neighbourhood of the zero matrix (1) $f(A+H)-f(A)$ is expressible in the form $\sum P_iHQ_i$, (2) $\lim_{H\to 0}\sum P_iQ_i$ exists and is independent of the manner of approach of H to 0. If (1) and (2) are satisfied then $\lim_{H\to 0}\sum P_iQ_i$ is called the derivative of $f(Z)$ at $Z=A$ and is denoted by $f^I(A)$.

This derivative if it exists is unique and satisfies desirable formulae for sums and products. Further, if $f(z)$ is a single valued scalar function of a complex variable then $f^I(A)$ exists if and only if $f(z)$ is analytic at each characteristic root of A. In this case $f^I(A)=f'(A)$.

D. E. Rutherford (St. Andrews).

Gaier, Dieter. Über ein Iterationsverfahren von Komatu zur konformen Abbildung von Ringgebieten. Z. Angew. Math. Mech. **36** (1956), 252–253.

Komatu [Proc. Japan Acad. **21** (1945), 146–155; MR **11**, 341] has given an iteration method for proving the existence of a canonical conformal mapping of a doubly-connected domain bounded by Jordan curves on a circular ring. The author indicates an explicit estimate for the rate of convergence of the process which implies in particular uniform convergence of the sequence of approximating functions in the entire domain. *J. A. Jenkins.*

Renggli, Heinz. Zur konformen Abbildung auf Normalgebiete. Comment. Math. Helv. **31** (1956), 5–40.

The author introduces a concept of normal domain for plane domains of infinite connectivity generalizing the concept of minimal slit domain due to Koebe. The complete definitions are too complicated to reproduce here. For suitable classes of domains he proves the existence of conformal mapping onto normal domains. Extensive use is made of the concept of extremal length. A similar but more special problem has recently been treated by Strebel [Ann. Acad. Sci. Fenn. Ser. A. I. no. **179** (1955); MR **16**, 917]. *J. A. Jenkins* (Notre Dame, Ind.).

Batyrev, A. V. Approximate conformal mapping of polygonal domains. Rostov. Gos. Ped. Inst. Uč. Zap. no. 3 (1955), 39–43. (Russian)

The problem of mapping $|t|>1$ conformally on the outside of a polygon is solved by a Schwarz-Christoffel formula. The author suggests expansion of the integrand in powers of $1/t$ as a means of obtaining approximate formulae for the mapping function. *W. H. J. Fuchs.*

Avdeev, N. Ya. Application of conformal mapping to the solution of certain boundary problems. Rostov. Gos. Ped. Inst. Uč. Zap. no. 3 (1955), 71–88. (Russian)

Expository article on the application of complex variable methods to the solution of two-dimensional potential problems. *W. H. J. Fuchs* (Ithaca, N.Y.).

Sapondžyan, O. M. On the expansion of the mapping function of Christoffel-Schwarz. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki **9** (1956), no. 9, 97–104. (Russian. Armenian summary)

Set

$$A\int_0^\zeta \prod_{s=1}^n (a_s-t)^{\alpha_s-1}\,dt=C\sum_{k=1}^\infty \frac{C_k}{k}\zeta^k \quad (C_1=1).$$

The author obtains a recursion formula which gives C_n in terms of $C_1, C_2, \cdots, C_{n-1}$. As illustrations he obtains explicitly power series for the functions which map $|\zeta|<1$ conformally onto the interior of (a) a rhombus and (b) a rectangle.

Similar formulas are obtained for the mapping of $|\zeta|<1$ onto the exterior of a polygon. *A. W. Goodman.*

Wagner, Richard. Ein Kontaktproblem der konformen Abbildung. J. Reine Angew. Math. **196** (1956), 99–132.

The author makes a contribution to the question of canonical conformal mapping on circular domains for domains of infinite connectivity by treating the existence and uniqueness problems for such mappings in the case of domains invariant under a doubly-periodic family of transformations when there are only a finite number of non-equivalent boundary continua. He also treats a similar problem for contact domains. *J. A. Jenkins.*

af Hällström, Gunnar. Zur Beziehung zwischen den Automorphiefunktionen und dem Flächentypus. Acta Acad. Abo. **20** (1956), no. 10, 12 pp.

A meromorphic function $w=f(z)$ maps a region of the z-plane conformally onto a covering surface R of the w-plane. A natural problem here is to determine the substitutions $S(z)$ with the property $f(S(z))\equiv f(z)$. If the projections of the branch points of R are finite in number, then R can be topologically represented by a line complex. The author has introduced a method for using the line complex in the study of $S(z)$ [C. R. Dixième Congrès Math. Scandinaves, 1946, Gjellerup, Copenhagen, 1947, pp. 97–107; MR **9**, 233]. The purpose of the present paper is to draw conclusions also about the conformal type of R.

The first result in this direction is due to Shimizu, who showed that each singlevalued $S(z)$ is linear if $f(z)$ is meromorphic in the whole finite plane [Jap. J. Math. **8**, 175–236 (1931), 237–304 (1932)]. Using the author's method, Fourès showed that there exists a hyperbolic surface such that $f(z)$ possesses a nonlinear singlevalued $S(z)$ [Ann. Sci. Ecole Norm. Sup. (3) **69** (1952), 183–201; MR **14**, 550]. The author gives a somewhat simplified alternative to Fourès' example. He also exhibits surfaces with an n-valent singlevalued $S(z)$ for arbitrary n. All these surfaces are simply connected and by necessity hyperbolic.

The existence of simply connected surfaces with every $S(z)$ n-valued is discussed. The author has established the existence of such parabolic and elliptic surfaces for $n=2$, 3, 4, and 6 [Acta Acad. Abo. **16**, no. 2 (1948), no. 4 (1949); MR **10**, 698]. He now conjectures that no other values are possible for these types, while for the hyperbolic type he constructs surfaces corresponding to an arbitrary n.

The method is shown to work even for infinite connectivity: a surface given by the reviewer to exemplify the class O_{AD} [Ann. Acad. Sci. Fenn. Ser. A. I. no. **50** (1948); MR **10**, 365] turns out to be hyperbolic.

L. Sario (Los Angeles, Calif.).

Tietz, Horst. Faber-Theorie auf nicht-kompakten Riemannschen Flächen. Math. Ann. **132** (1957), 412–429.

The author has previously investigated Faber expansions on closed Riemann surfaces [J. Reine Angew. Math. **190** (1952), 22–33; Math. Ann. **129** (1955), 44–49, 431–450; MR **13**, 833; **16**, 1012; **17**, 251; see also H. Röhrl, Arch. Math. **3** (1952), 93–102; MR **14**, 154]. In this paper he extends the theory to open Riemann surfaces. The reasoning is developed ab ovo, and can be read without reference to previous papers.

The basic tool is the following analogue of the classical Laurent decomposition of a holomorphic function into two integrals. Let R be an arbitrary open Riemann surface and $G\subset R$ a boundary neighborhood relatively bounded by an analytic Jordan curve C. The complement

$$G^*=R-G-C$$

is assumed to be relatively compact. The usual $d\zeta/(\zeta-z)$ is replaced by a differential $dF(\zeta, z)$ holomorphic with respect to ζ on $R-z$ and with a simple pole at z. Given a holomorphic function f on C, the operations

$$Lf=\frac{1}{2\pi i}\int_C f(\zeta)dF(\zeta, z), \quad L^*f=\frac{1}{2\pi i}\int_{-C} f(\zeta)dF(\zeta, z)$$

produce holomorphic functions of z on G and G^* respectively, with $f=Lf+L^*f$ on C. The natural analogue of a Cauchy function is a holomorphic function f on $G\cup C$ with the property $Lf\equiv f$. The problem is to expand such Cauchy functions into Faber series.

Map $G^*\cup C$ conformally onto a covering surface of the τ-plane such that C goes into the circle $|\tau|=1$. The Faber functions on G are defined as $\Phi_n=L\tau^{-n}$ for all integers n. The function f is developed in the classical Laurent series $\sum_{-\infty}^{+\infty}a_n\tau^{-n}$ in an annulus with inner contour $|\tau|=1$. The main result of the paper is that these coefficients a_n give the Faber expansion $\sum a_n\Phi_n$ of f in G.

Similar results are derived for differentials and for more general configurations C and G^*. A number of classical theorems on Faber expansions remain valid on open Riemann surfaces. *L. Sario.*

Lewy, H. On linear difference-differential equations with constant coefficients. J. Math. Mech. **6** (1957), 91–107.

This investigation is concerned with functions of a complex variable defined on a logarithmic Riemann surface with the origin and infinity as branch points. Denote by z^+ and z^- the points related to z by

$$\log z^+=\log z+2\pi i;\ \log z^-=\log z-2\pi i.$$

The equation under investigation is

$$(1)\qquad \frac{d}{dz}\,[F(z^+)-F(z)]+i[F(z^+)+F(z)]=0,$$

a difference-differential equation which arises in the study of gravity waves. The author studies solutions of (1) which are (a) continuous at the origin or (b) vanish at infinity with the aid of function-theoretic methods.

A. E. Heins (Pittsburgh, Pa.).

Jain, Mahendra Kumar. On the maximum real part of an integral function and its derivatives. Ganita **5** (1954), 203–214 (1955).

The author proves more than forty inequalities and limit relations. Most of these involve $A(r)=A_0(r)$ or $A_s(r)$, the maximum of Re $f^{(s)}(z)$ on $|z|=r$. Almost all of them are obtained by substituting $A_s(r)$ for

$$M_s(r)=\max|f^{(s)}(re^{i\theta})|\ (s\geqq 0)$$

in relations proved by S. M. Shah [see MR **4**, 137; **6**, 206; **9**, 342; **10**, 289; **12**, 16, 249; **14**, 366]. For example, if f is an entire function of order ρ, lower order λ, rank of maximum term on $|z|=r$ equal to $\nu(r)$ then

$$\lim\begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\log\{rA_1(r)/A(r)\}}{\log r}=\begin{matrix}\rho\\ \lambda\end{matrix},$$

and $\lim\inf A_{s+1}(r)/A_s(r)\leqq\lim\inf \nu(r)/r\leqq\lim\sup \nu(r)/r\leqq \lim\sup A_{s+1}(r)/A_s(r)$. *J. Korevaar* (Madison, Wis.).

Erdös, P.; and Kövári, T. On the maximum modulus of entire functions. Acta Math. Acad. Sci. Hungar. 7 (1956), 305–317. (Russian summary)

Let $M(r)$ be the maximum modulus function of an entire function. The authors show that there exists a power series $N(r)=\sum c_n r^n$ with $c_n \geqq 0$ such that

$$1/6 < M(r)/N(r) < 3.$$

The constants are not best possible, but it is not necessarily true that there is an $N(r)$ such that $M(r) \sim N(r)$: in fact, there exist an absolute constant $E>1.005$ and a maximum modulus function $M(r)$ such that for every $N(r)$ the inequality $1/E < M(r)/N(r) < E$ fails for arbitrarily large r. *R. P. Boas, Jr.* (Evanston, Ill.).

Kjellberg, Bo. A note on a problem of Boas. Ark. Mat. 3 (1957), 295–299.

The problem is whether if $f(z)$ is regular and of exponential type c for $x \geqq 0$, if $\lambda_n \uparrow \infty$ and $\lambda_{n+1}-\lambda_n \geqq 2\delta > 0$, and if ϕ is a nondecreasing convex function of $\log t$ with $\phi(0)=0$, then $\int_0^\infty \phi\{|f(x)|\}dx < \infty$ implies $\sum \phi\{e^{-c\delta}|f(\lambda_n)|\} < \infty$. The author shows that the answer is yes by first proving it when $\int_0^{\frac{1}{2}} (-t \log t)^{-1}\phi(t)dt < \infty$, and then showing that this apparently natural additional condition is redundant by proving that there are no functions for which $\int \varphi\{|f|\}dx < \infty$ while the last integral diverges. *R. P. Boas, Jr.*

Boas, R. P., Jr. Inequalities for asymmetric entire functions. Illinois J. Math. 1 (1957), 94–97.

Let f be an entire function of exponential type τ, with $|f(x)| \leqq 1$ for real x. Then one has the classica linequalities $|f(x+iy)| \leqq \exp(\tau|y|)$ and $|f'(x)| \leqq \tau$ of which the second one is due to S. Bernstein [for references see Boas, Entire functions, Academic Press, New York, 1954; MR 16, 914]. The author makes two additional assumptions, viz. that $\lim \sup_{y \to \infty} y^{-1} \log |f(iy)|=0$ and that $f(x+iy) \neq 0$ for $y>0$. He then proves that for $y<0$, $|f(x+iy)| \leqq \frac{1}{2}(e^{\tau|y|}+1)$, while $|f'(x)| \leqq \frac{1}{2}\tau$. The latter inequality generalizes a result on polynomials without zeros in the unit disc which was conjectured by Erdös and proved by Lax [Bull. Amer. Math. Soc. 50 (1944) 509–513; MR 6, 61]. *J. Korevaar.*

Bose, S. K. A note concerning some properties of the maximum function of a meromorphic function. Math. Z. 66 (1957), 487–489.

The author attempted to prove three theorems on the growth of meromorphic functions in a previous paper [Math. Z. 56 (1952), 223–226; MR 15, 23] which were subsequently shown to be incorrect by Shah and Singh [Math. Student 22 (1954), 121–128; MR 16, 459]. The author now attempts to prove his results under weaker conditions. The reviewer is unable to follow his arguments but believes that theorems A and B are now true. Theorem C is based on a false result wrongly attributed to the reviewer, but if lim is replaced by sup lim, theorem C is probably also true. *W. K. Hayman* (London).

Heins, Maurice. Asymptotic spots of entire and meromorphic functions. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 883–885.

Cette note annonce (sans démonstrations) d'importantes relations liant la croissance des fonctions entières et méromorphes à la notion de zone asymptotique (asymptotic spot) antérieurement introduite par l'auteur [Ann. of Math. (2) 61 (1955), 440–473; p. 457; MR 16, 1011].

A chaque zone asymptotique l'auteur associe un indice entier $\geqq 0$, l'indice harmonique qui permet de préciser le théorème de Denjoy-Carleman-Ahlfors.

Une extension naturelle de la notion de valeur omise conduit l'auteur à des résultats très précis concernant leur répartition et en particulier à un théorème du type de Picard. *L. Fourès* (Marseille).

Constantinescu, Corneliu. Einige Anwendungen des hyperbolischen Masses. Math. Nachr. 15 (1956), 155–172.

Several estimates are derived for the following three types of classical problems: 1) covering properties in the style of Bloch-Landau, 2) the Picard-Landau theorem, 3) "cercles de remplissage" in the sense of Julia-Milloux. Most of the results are so technical that they cannot be quoted here. One is a refinement of a theorem due to Bermant [Mat. Sb. N.S. 15(57) (1944), 285–324; MR 7, 150]. All proofs use the principle of hyperbolic measure.

L. Ahlfors (Cambridge, Mass.).

Collingwood, E. F.; and Lohwater, A. J. Applications of the theory of cluster sets to a class of meromorphic functions. Proc. Cambridge Philos. Soc. 53 (1957), 93–105.

Let $f(z)$ be meromorphic in $|z|<1$ and let a be an arbitrary complex number. Denote by $G[a, \sigma]$ the open set of all values z in $|z|<1$ for which $|f(z)-a|<\sigma$, where σ is a given positive number; a component $G_n[a, \sigma]$ of $G[a, \sigma]$ is called bounded if the closure $\bar{G}_n[a, \sigma]$ of $G_n[a, \sigma]$ is contained in $|z|<1$, otherwise $G_n[a, \sigma]$ is called unbounded. If α is an asymptotic value of $f(z)$ along a path terminating in a single point of $|z|=1$, then α is said to belong to $\Gamma_P(f)$. The main purpose of this paper is to extend a theorem of Collingwood concerning inequalities between deficiences of a given value of a meromorphic function in $|z|<\infty$ (parabolic case) and the valency of the function in certain domains bounded by level curves [Trans. Amer. Math. Soc. 66 (1949), 308–346; MR 11, 94] to the hyperbolic case. For that purpose, the following extension of Iversen's lemma plays an important rôle: Let $f(z)$ be meromorphic in $|z|<1$, and let $\Gamma_P(f)$ be of logarithmic capacity zero. Suppose that $f(z)$ takes no value more than p $(<\infty)$ times in some unbounded component $G_0[a, \sigma]$ of the set $G[a, \sigma]$ $(\sigma>0)$. Then, for every positive $\sigma_1<\sigma$ the components of the set $G[a, \sigma_1]$ contained in $G_0[a, \sigma]$ do not exceed p in number and are all bounded. *K. Noshiro.*

Clunie, J. On functions meromorphic in the unit circle. J. London Math. Soc. 32 (1957), 65–67.

If $f(z)$ is meromorphic in $|z|<1$, has a total of N zeros poles and unities in $|z|<1$ but none in $|z|<\delta$, then

$$T(\tfrac{1}{2}, f) < \alpha N \log(e/\delta)+\beta \log^+|1/f'(0)|+\gamma \log^+|f(0)|+C.$$

The present paper gives the values $\alpha=\beta=\frac{1}{4}\gamma=1/(1-\varepsilon)$ in place of Valiron's [Acta Math. 52 (1928), 67–92] estimates $\alpha=36$, $\beta=4$, $\gamma=12$. The proof employs a lemma recently introduced by Hayman [ibid. 86(1951), 89–191, 193–257; MR 13, 546]. *A. J. Macintyre* (Aberdeen).

Perron, Oskar. Über eine Schlichtheitsschranke von James S. Thale. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. 1956, 233–236 (1957).

Thale [Proc. Amer. Math. Soc. 7 (1956), 232–244; MR 17, 1063] showed that the function F defined, for $|x|<1$, by $F(x)=1/1+a_1x/1+a_2x/1+\cdots$, where each coefficient a_p in this continued fraction is a complex number with

modulus not greater than 1, is univalent for

$$|x|<12\sqrt{2}-16\doteq .97$$

and conjectured that F is univalent for $|x|<1$. The author shows that this conjecture is false and that $12\sqrt{2}-16$ is the largest number c such that every such F is univalent for $|x|<c$. *H. S. Wall* (Austin, Tex.).

Jørgensen, Vilhelm. On an inequality for the hyperbolic measure and its applications in the theory of functions. Math. Scand. 4 (1956), 113–124.

Let D be a domain in z-sphere, the $z=x+iy$, with at least three boundary points and $\lambda(z)$ the hyperbolic measure factor in D. The author proves that if D contains the half-plane $y<0$ and if z lies in D and the half-plane $y>0$ then $\lambda(z)\geqq\lambda(\bar{z})$ equality occurring only if the boundary of D lies on the real axis. The result is obtained by considering the equation $\Delta u=4e^{2u}$. The author gives a number of applications, obtaining, in particular, generalizations of theorems of Walsh [Amer. Math. Monthly 46 (1939), 472–485; MR 1, 111] and Ullman [Proc. Amer. Math. Soc. 2 (1951), 654–657; MR 13, 223]. He also gives simple direct proofs of these theorems. *J. A. Jenkins.*

Opial, Z. Sur une famille de fonctions analytiques. Ann. Polon. Math. 3 (1957), 312–318.

Let C denote family of functions f which are analytic in the unit disk K and for which (1) $\iint_K \ln^+|f(z)|d\sigma<\infty$. The example $\exp(1-z)^{-4/3}$ shows that Nevanlinna's class A is a proper subset of C. If $f\in C$ has the zeros a_j in K, and if $f\not\equiv 0$, then $\sum(1-|a_j|)^2<\infty$. Any subclass of functions in C for which the integral in (1) is uniformly bounded is a normal family. This conclusion holds also if in (1) the unit disk is replaced by an arbitrary domain D, and the operator $\ln^+$ by an operator ϕ such that $\phi(|f(z)|)$ is real, nonnegative, continuous and subharmonic in D whenever f is analytic in D, and unbounded if f is unbounded. *G. Piranian* (Ann Arbor, Mich.).

See also: Geronimus p. 879; Fox, p. 885; Sunouchi, p. 889; Inozemcev, p. 891; Tillmann, p. 910; Gel'fand, p. 913; Stein, p. 933; Gunning, p. 933; Dalla Volta, p. 934; Davis, p. 938; Fel'zenbaum, p. 965; Jones, p. 970.

Functions with Particular Properties

Ivanov, V. K. The inverse problem of potential for a body closely approximated by a given body. Izv. Akad. Nauk SSSR. Ser. Mat. 20 (1956), 793–818. (Russian)

Verf. beschäftigt sich mit dem Problem, die Gestalt eines mit Masse konstanter Dichte erfüllten Körpers T_1 zu bestimmen, dessen Raumpotential im Außengebiet bekannt ist. Er beschränkt sich dabei auf den Fall, daß die gegebene Potentialfunktion sich im Sinne einer gewissen Funktionalmetrik hinreichend wenig von dem Potential eines bekannten Körpers T gleicher Dichte unterscheidet; T ist bezüglich eines gewissen inneren Punktes sternförmig. Ähnlich wie bei Lichtenstein [z.B. Gleichgewichtsfiguren rotierender Flüssigkeiten, Springer, Berlin, 1933] wird die Berandung S_1 von T_1 in der Gestalt $v=\rho(\xi,\eta)$ angesetzt, wo ξ,η krummlinige Koordinaten auf der Randfläche S von T sind. S_1 soll in der Nachbarschaft von S, das hinreichend glatt vorausgesetzt wird, liegen; v bezeichnet den Abstand entsprechender, d.h. auf der gleichen Normalen zu S gelegener, Punkte auf S und S_1. Durch Entwicklung der Differenz der Normalableitungen der beiden Potentialfunktionen in entsprechenden Punkten auf S und S_1 in eine Art Integralpotenzreihe bezüglich der Funktion ρ und ihren Ableitungen wird eine Integro-Differentialgleichung für ρ erhalten, deren Lösung mittels sukzessiver Näherungen gelingt. Der teilweise komplizierte Nachweis der Konvergenz sowie der Unität bedient sich vielfach der Lichtensteinschen Methoden. *K. Maruhn* (Dresden).

Zahorska, H. Charakterisierung der Menge von Nichtexistenzpunkten des Randwertes harmonischer beschränkter Funktionen. Fund. Math. 43 (1956), 338–357.

Given R-integrable boundary values on the circumference of the unit circle, the author discusses the set N of boundary points where (for radial approach) the harmonic function determined by Poisson's integral fails to approach a limit. He gives as a characterization of N the following property: $N=\sum_{k=1}^{\infty}N_k$, where $N_k\in G_\delta$, $|N_k|=0$ and $N_k\cdot N_l=0$ for $k\neq l$. Here G_δ represents the class of points sets which are the products of a finite or denumerably infinite number of open sets and $|N_k|$ represents the measure of the closure of N_k. *F. W. Perkins.*

Tsuji, Masatsugu. On a non-negative subharmonic function in a half-plane. Kōdai Math. Sem. Rep. 8 (1956), 134–141.

Let $u(z)=u(x+iy)$ be a non-negative subharmonic function in the half-plane $x>0$ which vanishes continuously on the imaginary axis. If we set

$$m(r)=\int_{-\pi/2}^{\pi/2}u(re^{i\theta})\cos\theta\,d\theta$$

it is known that $m(r)/r$ is a non-decreasing function of r. The author adds the information that it is a convex function of $1/r^2$.

If $c=\lim_{r\to\infty}m(r)/r$ is finite he derives the representation

$$u=\frac{2cx}{\pi}-\int_{\operatorname{Re}a>0}\log\left|\frac{z+\bar{a}}{z+a}\right|d\mu(a),$$

where the positive mass-distribution μ satisfies the condition

$$\int_{\operatorname{Re}a>0}\frac{\operatorname{Re}a}{|a|^2}\,d\mu(a)<\infty.$$

From this representation a simple proof is obtained for a result of Ahlfors and Heins [Ann. of Math. (2) 50 (1949), 341–346; MR 10, 522], namely that $\lim_{r\to\infty}u(re^{i\theta})/r=(Lc/\pi)\cos\theta$ except for a set of θ of capacity zero. *L. Ahlfors.*

Fox, William C. The critical points of Peano-interior functions defined on 2-manifolds. Trans. Amer. Math. Soc. 83 (1956), 338–370.

Let M be a 2-manifold with boundary (which, by definition, may be without a boundary). A continuous real valued function f is called Peano interior if it is interior (i.e., maps open sets on open sets) and every level curve of f is locally connected at every one of its points which is interior to M (and is therefore a Peano space). The class of Peano interior functions is a proper extension of the class of pseudo-harmonic functions, i.e., functions which are topologically equivalent to harmonic functions [for the exact definition, see M. Morse, Topological methods in the theory of functions of a complex variable, Princeton, 1947; MR 9, 20.]

The first goal of the author is the proof of the "spoke theorem" which constitutes a generalization of a well known theorem for harmonic functions. It is proved for boundary points with a certain property as well as for interior points p of M. For the latter, it may be formulated as follows: let $M^p=\{x\subset M|(fx)=f(p)\}$. The point p may be a component of M^p. If that is not the case, then there exists an arbitrarily small simply connected domain N containing p with compact closure $\bar{N}$ of the property: If C is a component (or "spoke") of $(\bar{N}\cap M^p)-(p)$, then $\bar{C}$ is an arc whose interior is in N and whose boundary consists of p and a boundary point of N, the number of spokes being an even number $2m$. The following necessary and sufficient condition for a real valued continuous function f defined on a 2-manifold without boundary to be Peano interior is a corollary to the spoke theorem: each level curve of f is a locally finite linear graph.

For interior points p the spoke theorem makes possible the following definitions: the order of p with respect to f is the number $2m$ of spokes; the number $m-1$ is called the multiplicity of p. If $m-1>0$, p is called a critical point of f. The main result of the paper is the following theorem: let M be a compact orientable 2 manifold with n boundary curves and of genus g. Let f be a Peano interior function such that each boundary curve is a level curve of f. Then the sum of the multiplicities is $2g+n-2$ (in other words, it is one less than the first Betti number of M). As the author remarks, this result may be considered as an extension of a theorem stated by F. Klein [Über Riemann's Theorie der algebraischen Funktionen und ihrer Integrale, Leipzig, Teubner 1882] for harmonic functions with n logarithmic poles on a compact Riemann surface without boundary of genus g. In a second paper the author intends to treat the case where f is not necessarily constant on each of the boundary curves.

E. H. Rothe (Ann Arbor, Mich.).

Kovan'ko, A. S. On a cer tain property and a new definition of generalized aln ost periodic functions of A. S. Bezikovič. Ukıain. Mat. Ž. 8 (1956), 273–288. (Russian)

R. Doss [Ann. of Math. (2) **59** (1954), 477–489; MR **16**, 242] has defined a new class of almost periodic functions and has shown it to be identical with the class (B_p a.p.) [Besicovitch, Acta Math **58** (1932), 217–230]. The author has found a new property of B_p a.p. functions which he uses to obtain Doss' result in a simpler fashion. Ideas involved are characterized by the following definitions. The measure δ of a set, defined by means of Lebesgue measure is given in the following way:

$$\delta E=\limsup_{T\to\infty} \text{meas}\,[E\cap -T,T)]/2T.$$

We shall say $f\in A_p$, if (i) f is a B_p uniformly measurable function. (ii) To every $\varepsilon>0$ there exists an $\eta>0$ and a relatively dense set of ε-periods τ, such that for all x, and all $\tau-\eta<t<\tau+\eta$, $|f(x+t)-f(x)|<\varepsilon$ may fail only in a set of t of $\bar{\delta}$-measure $<\varepsilon$. (iii) Finally, for every $a>0$, $\lim (1/n)\sum_{k=0}^{n} f(x+na)$ converges in measure to a function $f^{(a)}$, almost everywhere finite and of period a.

The main result of the paper is that the classes B_p a.p. and A_p are identical. *František Wolf.*

Bredihina, A. A. On the absolute convergence of Fourier series of almost periodic functions. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 1163–1166. (Russian)

Generalising Sidon's Theorem on gap Fourier series the author proves the following theorem. If $\sum_{k=-\infty}^{\infty} A_k e^{i\lambda_k x}$ ($-\lambda_{-k}=\lambda_k>0$ for $k>0$) is the Fourier expansion of the almost periodic function $f(x)$ and if $\lambda_{k+1}/\lambda_k>\theta>1$ ($k\geqq 1$), then $\sum_{-\infty}^{\infty}|A_k|<C(\theta)\sup|f(x)|$. A theorem for the case $\lambda_{k+1}/\lambda_k<\alpha<1$ is also proved. If $\{\mu_k\}_{k=-\infty}^{\infty}$ is a sequence with $-\mu_{-k}=\mu_k$ for $k>0$, $\mu_k\uparrow\infty$ ($k>0$) of which the sequence $\{\lambda_k\}$ of the theorem is a sub-sequence, then the best approximation of $f(x)$ by trigonometric polynomials with exponents μ_k, $|\mu_k|\leqq n$, is of the order of magnitude $\sum_{|\lambda_k|>n}|A_k|$. *W. H. J. Fuchs* (Ithaca, N.Y.).

Kupcov, N. P. On absolute and uniform convergence of Fourier series of almost periodic functions. Mat. Sb. N.S. **40(82)** (1956), 157–178. (Russian)

The author considers the absolute and uniform convergence of Fourier series of uniformly almost periodic functions f whose exponents $\lambda_1, \lambda_2, \cdots$ satisfy one of the following two conditions: a) $\{\lambda_n\}$ has only a finite number f limit points $b_1, b_2, \cdots, b_s$, none of which coincides with . ny λ; b) there is a $C>0$ such that $\lambda_n>Cn$ for all n. Let $M\{f\}=\lim (2T)^{-1}\int_{-T}^{T} f\,dx$. The following two results are typical. 1) In case a), if there is a $C>0$ such that for each n there is a j satisfying the condition $|\lambda_n-b_j|\leqq C/n$, and if the series $\sum\{n^{-1}\omega(n^{-1})\}^{\frac{1}{2}}$ converges, where

$$\omega(\delta)=\max_{j=1,2,\cdots,s}\delta^2 M_x\{|\int_0^\infty e^{-(\delta-ib_j)}f(x-t)\,dt|^2\},$$

then the Fourier series of f converges absolutely. 2) In case b), if $\sum n^{-\frac{1}{2}}\omega(1/n)<\infty$, where

$$\omega(\delta)=\sup_{|h|\leqq\delta}(M_x\{|f(x+h)-f(x)|^2)^{\frac{1}{2}},$$

the Fourier series of f converges absolutely.

A. Zygmund (Chicago, Ill.).

Gončar, A. A. On a new quasi-analytic class of functions. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 930–932. (Russian)

The author considers quasi-analytic classes analogous to those he considered before [same Dokl. (N.S.) **100** (1955), 205–208; MR **16**, 803] but now requiring only that the functions are measurable. Let $R_n(f;F)$ be the best approximation to f (continuous on the closed set F) by rational functions of order at most n. Let R consist of those measurable functions on [0, 1] such that for each $\varepsilon>0$ there is a closed F_ε of measure exceeding $1-\varepsilon$ on which $\{R_n(f;F_\varepsilon)\}^{1/n}\to 0$. Then two elements of R which coincide on a set of positive measure coincide almost everywhere. Any element of R has almost everywhere an asymptotic derivative that is, an approximate derivative in Denjoy's terminology; cf. Lusin, Integral and trigonometric series, Gostehizdat, Moscow-Leningrad, 1951, p. 445, note 87; MR **14**, 2] belonging to R. If $R_n(f;F_\varepsilon)\leqq C(\varepsilon)N^{-p-\delta}$, then f has almost everywhere a pth asymptotic derivative. *R. P. Boas, Jr.*

See also: Hummel, p. 878; Avdeev, p. 882; Vekilov, p. 903; Doob, p. 941; Lenz, p. 968.

Special Functions

Al-Salam, W. A. Some relations involving the Jacobi polynomials. Portugal. Math. **15** (1956), 73–77.

L'auteur propose ici des généralisations du développement de $(1-x)^k$ en série de polynomes de Jacobi; de la

formule connue:

$$(1-x)^k=2^k\Gamma(\alpha+k+1)$$
$$\times \sum_{s=0}^{k} \frac{(-k)_s(2s+\alpha+\beta+1)\Gamma(s+\alpha+\beta+1)}{\Gamma(s+\alpha+1)\Gamma(s+\alpha+\beta+k+2)} P_s^{\alpha,\beta}(x),$$

il tire par des opérations simples:

$$(x-y)^k=2^k k! \sum_{s=0}^{k} \frac{(2s+\alpha+\beta+1)\Gamma(s+\alpha+\beta+1)}{\Gamma(s+k+\alpha+\beta+2)}$$
$$\times P_s^{\alpha,\beta}(x)P_{k-s}^{(-k-\alpha-s,-k-\beta-1)}(y).$$

Il dérive alors de là d'autres formules fournissant plusieurs expressions d'un polynome de Jacobi $P_n^{\alpha,\beta}(x)$ sous forme d'une somme de n termes constitués eux-mêmes par des polynomes de Jacobi. *R. Campbell* (Caen).

af Hällström, Gunnar. Über halbvertauschbare Polynome. Acta Acad. Åbo. 21 (1957), no. 2, 20 pp.

The functions $g(z)$ and $h(z)$ are said to be semipermutable with respect to substitution as law of composition if $g(h(z))=L(h(g(z)))$ or more simply $gh=Lhg$, where L is a linear fractional function. After establishing a few general results about semipermutable functions the author confines attention to polynomials, when L necessarily becomes a linear polynomial. His investigations are to some extent analogous to those of E. Jacobsthal concerning permutable polynomials ($L=z$) [Math. Z. 63 (1955), 243–276; MR 17, 574].

A sequence of semipermutable functions $f_0, f_1, f_2, \cdots, f_n, \cdots$ in which f_n is of degree n, is called a chain. It is shown that, apart from similarity transformations, the only chains are (i) $f_n=\alpha_n z^n$, where α_n is a constant and (ii) $f_n=\pm T_n$, where T_n is the nth Chebyshev polynomial suitably normalized. There are numerous other results dealing with special cases or classes of polynomials. The paper concludes with a section on older function-theoretic investigations of J. Ritt and their bearing on the present problem. *W. Ledermann* (Manchester).

Kemeny, John G. The exponential function. Amer. Math. Monthly 64 (1957), 158–160.

The purpose of this paper is to discuss a slightly unorthodox definition of the exponential function. This definition allows us to derive the basic properties of e^x and $\log x$ by very elementary means.

Author's summary.

Simonart, Fernand. Sur l'adjointe de l'équation de Bessel. Acad. Roy. Belg. Bull. Cl. Sci. (5) 42 (1956), 1094–1101.

In the terminology of the author, Bessel's equation is

(A) $$xy''+2(n+1)y'+xy=0$$

which has the adjoint

(B) $$xY''-2nY'+xY=0.$$

Equation (B) may also be obtained from (A) by the substitution

$$Y=yx^{2n+1}.$$

Hence, for n non-negative integral, the solution of (B) is given in finite terms by a linear combination of

$$x^{2n+1}(D^2+1)^n\frac{\sin x}{x} \text{ and } x^{2n+1}(D^2+1)^n\frac{\cos x}{x},$$

in which $D=d/dx$. Remarkably, the author fails to mention the connection with Bessel functions of half-integral order. His definition of a_ν in section 2 should read

$$a_\nu=\frac{1}{\nu!(n+1)\cdots(n+\nu)}.$$

C. J. Bouwkamp (Eindhoven).

Nörlund, N. E. Sur les fonctions hypergéométriques d'ordre supérieur. Mat.-Fys. Skr. Danske Vid. Selsk. 1 (1956), no. 2, 47 pp.

This paper gives a very full, rigorous and classical treatment of some integrals from generalized hypergeometric function theory, based on the differential equation $Q(\vartheta)y-zR(\vartheta)y=0$, where $\vartheta\equiv zd/dz$ and $Q(z)$ and $R(z)$ are polynomials of the same degree. This equation has singularities at $z=0$, $z=1$ and $z=\infty$.

The introduction outlines the known theory and defines the solutions $y_s^*(z)$ and $\bar{y}_s^*(z)$, where

$$y_s^*(z)=$$
$$\prod_{\nu=1}^{n}\frac{\Gamma(\alpha_\nu+\gamma_s)}{\Gamma(\gamma_s-\gamma_\nu+1)}z^{\gamma_s} F\left(\begin{matrix}\alpha_1+\gamma_s & \cdots & \alpha_n+\gamma_s\\ \gamma_s-\gamma_1+1 & \cdots * \cdots & \gamma_s-\gamma_n+1\end{matrix}\middle| z\right)$$

(here * indicates that $\gamma_s-\gamma_s+1$ is omitted) and $\bar{y}_s^*(z)$ is obtained from $y_s^*(z)$ by writing $1/z$ for z and interchanging α and γ. Integrals are given for the functions $\xi_n(z)$ and $\bar{\xi}_n(z)$,

$$\int_0^1 z^{x-1}\xi_n(z)dz=\Gamma(\beta_n+1)\prod_{s=1}^{n}\frac{\Gamma(x+\gamma_s)}{\Gamma(x-\alpha_s+1)},$$

and

$$\int_1^\infty z^{x-1}\bar{\xi}_n(z)dz=\Gamma(\beta_n+1)\prod_{s=1}^{n}\frac{\Gamma(\alpha_s-x)}{\Gamma(1-\gamma_s-x)}.$$

The first chapter develops integrals of the type

$$\sum_{\nu=1}^{n}\frac{\Gamma(\alpha_\nu+\gamma)}{\Gamma(\gamma-\gamma_\nu+1)} F\left(\begin{matrix}1 & \alpha_1+\gamma\cdots\alpha_n+\gamma\\ & \gamma-\gamma_1+1\cdots\gamma-\gamma_n+1\end{matrix}\middle| z\right)$$
$$=\frac{1}{\Gamma(\beta_n+1)}\int_1^\infty t^{-\gamma}\bar{\xi}_n(t)\frac{dt}{t-z},$$

and their Mellin transforms, and deduces corresponding generalizations for $y_s^*(z)$ and $\bar{y}_s^*(z)$, together with a number of related results. The asymptotic forms of the expansions are stated, a number of special cases are discussed in detail and the logarithmic forms of solutions are also developed. Several of the results are special cases of the general theorems stated by the reviewer [Proc. Cambridge Philos. Soc. 51 (1955), 288–296, 577–589; MR 16, 1106; 17, 150].

The second chapter discusses functions and integrals of the type

$$\int_\infty^{(1-)} z^{x-1}(1-z)^{-x-\gamma_s}\xi_{n-1}\left(\begin{matrix}\alpha_1\cdots\alpha_{j-1} & \alpha_{j+1}\cdots\alpha_n\\ \gamma_1\cdots\gamma_{s-1} & \gamma_{s+1}\cdots\gamma_n\end{matrix}\middle| z\right)dz$$

and their Mellin transforms. Again the logarithmic forms are considered. The third chapter gives integrals of the type

$$\int_0^{(1+)} z^{x-1}(z-1)^{\alpha_s-x-1}\bar{\xi}_n(z)dz,$$

and their Mellin transforms, and the fourth chapter discusses solutions defined by

$$y_{s,r}(z)=\frac{\pi}{\sin\pi(\gamma_s-\gamma_r)}[y_r^*(z)-y_s^*(z)].$$

which are holomorphic about $z=1$. Again integral representations and Mellin transforms are given together with further asymptotic forms. *L. J. Slater.*

Macrobert, T. M. On recurrence formulae. Proc. Glasgow Math. Assoc. 3 (1956), 36–37.

Dans les formules de récurrence habituelles de la fonction hypergéométrique, par exemple dans celle-ci:

$$\gamma F(\alpha, \beta-1; \gamma; z)-\gamma F(\alpha-1, \beta; \gamma; z) +(\alpha-\beta)zF(\alpha, \beta; \gamma+1, z)=0,$$

l'auteur remplace α, β, γ, z, par α_1, α_2, ρ_1-1, $-1/z$, et par des modifications d'écriture; il écrit ainsi cette formule (avec la nouvelle notation E) sous la forme

$$(\alpha_1-\alpha_2)E(\alpha_1, \alpha_2; \rho_1, z)=(\alpha_2-1)zE(\alpha_1, \alpha_2-1\ \rho_1-1, z) -(\alpha_1-1)zE(\alpha_1-1, \alpha_2\ \rho_1-1, z)$$

qui permet d'écrire de nouvelles formules plus générales dont il déduit comme cas particuliers des formules de récurrence connues des fonctions de Bessel et de Legendre. *R. Campbell* (Caen).

See also: Ciorănescu, p. 876; Griffith, p. 895; Mehra, p. 896; Satake, p. 934; Sharma, p. 963.

Sequences, Series, Summability

Slater, L. J.; and Lakin, A. Two proofs of the ${}_6\Psi_6$ summation theorem. Proc. Edinburgh Math. Soc. (2) 9 (1956), 116–121.

Author's introduction: The ${}_6\Psi_6$ summation theorem was first proved by Bailey [Quart. J. Math. Oxford Ser. 7 (1936), 105–115], who deduced it indirectly from a transformation of a well-poised ${}_8\Phi_7$ series into two ${}_4\Phi_3$ series. No direct proof of the theorem has been published, and since it has interesting applications in the proofs of various identities which occur in combinatory analysis, for example the A series of Rogers and some elegant identities due to Ramanujan [see Bailey, ibid. (2) 3 (1952), 29–31; MR 13, 725], we give two new proofs of the theorem in this paper. The first proof (due to Slater) introduces a basic analogue of the Barnes type integral. The second (due to Lakin) is the basic analogue of an operational method used elsewhere [Burchnall and Lakin, ibid. 1 (1950), 161–164; MR 12, 178], and provides an application of Carlson's theorem. *N. J. Fine.*

Sierpiński, W. On certain expansions of real numbers into infinite fastconverging products. Prace Mat. 2 (1956), 131–138. (Polish. Russian and English summaries)

Theorems 1 and 2. To every real number $x>1$ and every sequence n_k of natural numbers there exists a unique expansion $x=(1+n_1/d_1)(1+n_2/d_2)\cdots$, where the d_k are natural numbers and satisfy

$$d_{k+1}>(d_k-1)(d_k+n_k)n_{k+1}/n_k.$$

Here x is rational if and only if

$$n_k(d_{k+1}-1)=(d_k-1)(d_k+n_k)n_{k+1}$$

for large k. This strengthens results of A. Oppenheim [Quart. J. Math. Oxford Ser. (2) 4 (1953), 303–307; MR 15, 608]. The special case $n_k=1$ (Theorem 3) is due to G. Cantor [Z. Math. Phys. 14 (1869), 152–158] and also to the author [Prace Mat.-Fiz. 20 (1909), 215–234]. The case $n_k=2$ (Theorem 4) has been treated by the author [Bull. Soc. Roy. Sci. Liège 22 (1953), 520–529; MR 15, 699] and, for $x=\sqrt{((k+2)/(k-2))}$ with k a natural number, by E. B. Escott [Amer. Math. Monthly 44 (1937), 644–646], who gives the expansion

$$\sqrt{((k+2)/(k-2))}=(1+2/(k_1-1))(1+2/(k_2-1))\cdots$$

with $k_1=k$ and $k_{n+1}=k_n(k_n^2-3)$. *K. Zeller.*

Jadraque, V. Martin. On some formulas for continued fractions. Gac. Mat., Madrid (1) 9 (1956), 148–154. (Spanish)

The author obtains three special cases of Euler's continued fraction equivalent to a given series. These formulas are applied to obtain some well known continued fraction expansions. *W. T. Scott* (Evanston, Ill.).

Carr, A. J. On the arithmetico-geometric series. Math. Gaz. 41 (1957), 44–46.

Ingenious elementary manipulation of the general arithmetico-geometric series yields results about the binomial theorem, the sum of homogeneous products, the binomial coefficients, and sums of powers of integers.

Tanaka, Chuji. On the singularities of Dirichlet series. Comment. Math. Helv. 31 (1957), 184–194.

Let $F(s)=\sum a_n \exp(-\lambda_m s)$ $(0\leqq\lambda_1<\lambda_2<\cdots<\lambda_m\to+\infty)$ be a Dirichlet series having a finite abscissa of convergence $\sigma=\sigma_s$. The objective of the present paper is to establish the following results. Under the above assumptions there exists a sequence $\{\varepsilon_m\}$ $(\varepsilon_m=\pm1)$ such that $\sum \varepsilon_m a_m \exp(-\lambda_m s)$ has $\sigma=\sigma_s$ as a natural boundary. Also under the same hypothesis there exists a new Dirichlet series $\sum b_m \exp(-\lambda_m s)$ having $\sigma=\sigma_s$ as the natural boundary and such that either

$$|b_m|=|a_m|\ (n=1, 2, \cdots),\ \lim_m |\arg b_m-\arg a_m|=0,$$

or

$$\arg(b_m)=\arg a_n\ (n=1, 2, \cdots),\ \lim_m |b_m/a_n|=1.$$

Under the additional assumptions that $\lim_n \log n/\lambda_n=0$ these two theorems were proven earlier by Szász [Math. Ann. 85 (1922), 99–110]. The method of proof is based upon a criterion of Ostrowski's for singular points.

V. F. Cowling (Lexington, Ky.).

Bajšanski, Bogdan M. Sur une classe générale de procédés de sommations du type d'Euler-Borel. Acad. Serbe Sci. Publ. Inst. Math. 10 (1956), 131–152.

Der Verf. untersucht die Permanenz von Matrixverfahren $(a_{n\nu})$, die aus Funktionen $f(z)$ durch

$$f^n(z)=\sum_{\nu=0}^{\infty} a_{n\nu}z^\nu$$

hervorgehen. (Spezialfälle: Euler-Knopp, Meyer-König, Karamata, Borel-Gaier). Die Hauptschwierigkeit besteht darin, Bedingungen über $f(z)$ zu finden, die

$$\sum_{\nu=0}^{\infty} |a_{n\nu}|=O(1)$$

nach sich ziehen. Verf. zeigt: Ist (i) $f(z)$ regulär für $|z|<R$, $R>1$, (ii) $|f(z)|<1$ für $|z|\leqq1$, $z\neq1$, (iii) $f(1)=1$, (iv) $f(z)-z^\alpha=Ai^p(z-1)^p+o(1)(z-1)^p$, $z\to1$, $\alpha=f'(1)$, $\Re A\neq0$, so ist $\sum_{\nu=0}^{\infty} |a_{n\nu}|=O(1)$ $(n\to\infty)$. Die Bedingung (iv) kann nicht aus (i), (ii), (iii) gefolgert werden, sie ist in jedem der folgenden Fälle gültig: 1) $\Re f''(1)\neq f'^2(1)-f'(1)$,

2) die $a_{1\nu}$ sind reell und $f(z)-z^\alpha$ verschwindet in $z=1$ von gerader Ordnung, und 3) die $a_{1\nu}$ sind nicht negativ. Zum Beweis werden die $s_{n\nu}$ mittels des Cauchyschen Integralsatzes über passende Wege (die von n und ν abhängen) ausgedrückt. Für die Verfahren von Karamata,

$$[\alpha+(1-\alpha-\beta)z]/(1-\beta z) \ (\alpha<1, \beta<1, \alpha+\beta>0),$$

werden mit Hilfe dieses Ergebnisses Permanenz- und Vergleichssätze beweisen. Schliesslich wird gezeigt, dass aus (i) und $|f(z)|=1$ für $|z|=1$, $f(z)\neq e^{\theta_k i}z^k$ ($k=0, 1, \cdots$) die Beziehung $\sum_{\nu=0}^{\infty}|a_{n\nu}|\to\infty$ folgt. *A. Peyerimhoff.*

Petersen, G. M. Almost convergence and two matrix limitation methods. Math. Z. **66** (1956), 225–227.

Es seien B und C die Matrixverfahren $\frac{1}{2}(s_{2m+1}+s_{2m+2})$ und $\frac{1}{2}(s_{2m}+s_{2m+1})$ ($m=0, 1, 2, \cdots$). Der Verf. zeigt: 1) Eine Folge ist genau dann fastkonvergent (d.h. $t_{n,p}=(s_n+\cdots+s_{n+p})/p\to S$ für $p\to\infty$ und gleichmässig in n [G. G. Lorentz, Acta Math. **80** (1948), 167–180; MR **10**, 367], wenn $\{s_n\}$ die Summe einer beschränkten B-limitierbaren und einer beschränkten C-limitierbaren Folge ist. 2) Es gibt kein Matrixverfahren, das nicht zeilenfinit ist und stärker als C oder mit C äquivalent ist. *A. Peyerimhoff* (Giessen).

Sunouchi, Gen-ichirô. Theorems on power series of the class H^p. Tôhoku Math. J. (2) 8 (1956), 125–146.

Let $f(z)=\sum_{n=0}^{\infty} c_n z^n$, be regular in $|z|<1$ ($z=re^{i\theta}$), and assume that $f(z)$ belongs to the class H^p ($0<p<\infty$). Let $\sigma_n^\alpha(\theta)$ denote the Cesàro means of order α of the series $\sum_{n=0}^{\infty} c_n e^{in\theta}$. Define

$$h_\alpha(\theta)=\Big(\sum_{n=1}^{\infty}\frac{|\sigma_n^{\alpha-1}(\theta)-\sigma_n^\alpha(\theta)|^2}{n}\Big)^{\frac{1}{2}},$$

$$g_\alpha^*(\theta)=\Big(\int_0^1(1-r)^{2\alpha}dr\int_0^{2\pi}\frac{|f'(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^{2\alpha}}d\varphi\Big)^{\frac{1}{2}}$$

The author then proves

$$(*)\qquad A_\alpha h_\alpha(\theta)\leqq g_\alpha^*(\theta)\leqq B_\alpha h_\alpha(\theta),$$

$$(\dagger)\qquad \int_0^{2\pi}\{g_\alpha^*(\theta)\}^p d\theta\leqq A_p\int_0^{2\pi}|f(e^{i\theta})|^p\log^+|f(e^{i\theta})|d\theta+A_p,$$

$$(\dagger\dagger)\qquad \int_0^{2\pi}\{g_\alpha^*(\theta)\}^{p\mu}d\theta\leqq B_{p,\mu}\Big(\int_0^{2\pi}(|f(e^{i\theta})|^p d\theta\Big)^{1/\mu},$$

where $0<p\leqq1$, $\alpha=1/p$, and $0<\mu<1$. By using (*) and (†) the author then deduces

$$\int_0^{2\pi}\{\sup_n|\sigma_n^\alpha(\theta)|\}^p d\theta\leqq B_p\int_0^{2\pi}|f(e^{i\theta})|^p\log^+|f(e^{i\theta})|d\theta+B_p,$$

whenever $0<p\leqq\frac{1}{2}$, and $\alpha=(1/p)-1$.

The proof of these theorems depends heavily on the results of Littlewood and Paley, and Zygmund. Results almost identical with the above were obtained independently by A. Zygmund [Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 208–212; MR **17**, 1080].

Another result obtained by the author is the proof of the following conjecture of Zygmund,

$$\int_0^{2\pi}\Big\{\sup_n\frac{|\sigma_n^\alpha(\theta)|}{(\log(n+2))^{1/p}}\Big\}^p d\theta\leqq B_p\int_0^{2\pi}|df(e^{i\theta})|^p d\theta,$$

for $0<p\leqq1$, $\alpha=(1/p)-1$.

The paper also contains other related results, and a review of previously known theorems of the theory. *E. M. Stein* (Cambridge, Mass.).

Shanks, E. Baylis. Convergence of series with positive terms. Amer. Math. Monthly **64** (1957), 338–341.

It is the purpose in what follows to prove necessary and sufficient conditions for convergence of series with positive terms that serve as a general framework for short proofs of the sufficient conditions of many of the known tests for convergence or divergence of such series. *From the introduction.*

Krzyż, J. Olivier's theorem and its generalizations. Prace Mat. 2 (1956), 159–164. (Polish. Russian and English summaries)

Verfasser bringt den Satz von Olivier [J. Reine Angew. Math. 2 (1827), 31–44: "Konvergiert $\sum a_n$ und gilt $a_n\downarrow0$, so auch $na_n\to0$."] sowie Verallgemeinerungen in Zusammenhang mit einem Ergebnis von Kronecker [C. R. Acad. Sci. Paris **103** (1886), 980–987: "Konvergiert $\sum a_n$ und gilt $p_n\uparrow\infty$, so auch $(p_1a_1+\cdots+p_na_n)/p_n\to0$."]. Weitere Aussagen vergleichen die Konvergenz von $\sum\mu_nb_n$ mit der von $\sum(\mu_1+\cdots+\mu_n)(b_n-b_{n+1})$ und werden mittels partieller Summation gewonnen. Er betrachtet eingehender gewisse Spezialfälle, etwa $p_n=n^\sigma$ oder $\mu_n=0$; 1 und $b_n=1/n$. *K. Zeller* (Tübingen).

Karadžić, Lazar. Généralisation de certains théorèmes de la théorie des séries à termes positifs. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 1110–1117.

If a_n is positive and bounded for $n=1, 2, \cdots$, $S_n=\sum_{m=1}^{n}a_m$, and $f(x)$ is positive and monotonic for $x>0$, the series

$$\sum a_{n+1}f(S_n) \text{ and } \sum(S_{q_{n+1}}-S_{q_n})f(S_{q_n}),$$

for a strictly increasing sequence of integers q_n, are equiconvergent if $(S_{q_n})^{1/n}$ is bounded.

In particular, if a_n decreases, the series

$$\sum a_n \text{ and } \sum(q_{n+1}-q_n)a_{q_n}$$

are equiconvergent if $(q_n)^{1/n}$ is bounded. If $\sum a_n$ diverges (a_n positive and bounded, but not necessarily decreasing), $\sum a_nf(S_n)$ and $\int^\infty f(x)dx$ are equiconvergent if $(S_n)^{1/n}$ is bounded. *H. R. Pitt* (Nottingham).

See also: Glatfeld, p. 874; Stoilow, p. 876; Henstock, p. 880; Sapondžyan, p. 882; van der Corput, p. 890; Talalyan, p. 891; Matsumoto, p. 892; Kanno, p. 892; Prasad and Bhatt, p. 892; Duffin, p. 893; Doetsch, p. 894; Heywood, p. 896; Kennedy, p. 896; Boas and Gonzalez-Fernândez, p. 896; Blackman, p. 896.

Approximations, Orthogonal Functions

Whitney, Hassler. On functions with bounded nth differences. J. Math. Pures Appl. (9) **36** (1957), 67–95.

Pour $f(x)$ réelle et continue définie dans un segment I, ou dans $I^*=[0, +\infty)$, ou dans $I^{**}=(-\infty, +\infty)$, on considère les différences d'ordre n:

$$\Delta_h^n f(x)=\sum_0^n(-1)^{n-i}\binom{n}{i}f(x+ih);$$

il existe une constante K_n et un polynome P de degré $<n$, tels que:

$$|f(x)-P(x)|\leqq K_n\sup_{h,y}|\Delta_h^n(y)|$$

lorsque f est définie dans I; lorsque f est définie dans I^* ou I^{**}, la constante K_n est à remplacer par une autre: K_n^* ou K_n^{**}. Le cas $n=1$ est bien connu, le cas $n=2$ a été traité par H. Burkill [Proc. London Math. Soc. (3) **2** (1952), 150–174; MR **14**, 162]. Pour n pair, l'auteur

donne la valeur exacte de la constante K_n^{**}; dans les autres cas, pour $n>2$, il donne seulement des inégalités obtenues à partir de cas particulier. Diverses autres constantes sont définies et utilisées. Le résultat est obtenu en utilisant un polynome interpolateur. *J. Favard.*

Ghéorghiev, Ghéorghi Iv. Formules de quadrature mécanique à nombre minimum de termes pour les intégrales triples. Univ. d'Etat Varna "Kiril Slavianobâlgarski" Fac. Tech. Constructions. Annuaire 3 (1947–1948), 97–123 (1949). (Bulgarian. French summary)

La classe C_n est composé de polynômes $\varphi(x, y, z)$ des variables réelles x, y, z dont le degré ne dépasse pas le nombre naturel n. R est un domaine dans l'espace des variables x, y, z tel que les intégrales R est un domaine dans l'espace des variables x, y, z tel que les intégrales triples

$$I_{klm}=\iiint_{(R)} x^k y^l z^m dxdydz \ (k, l, m=0,1, \cdots, n;\ k+l+m\leq n)$$

existent. L'auteur cherche à déterminer une formule

$$(1)\qquad \iiint_R \varphi(x, y, z)dxdydz=\sum_{m=1}^{n} \lambda_m\varphi(x_m, y_m, z_m)$$

exacte pour chaque polynôme $\varphi(x, y, z)$ de la classe C_n, pour laquelle le nombre N des termes au second membre est le plus petit possible. Il détermine effectivement toutes les formules de cette espèce pour les classe C_1 et C_2.

Pour le cas C_1 le nombre minimum des termes est égal à 1. Il existe une seule formule de la forme (1), qui est exacte pour chaque polynôme de la classe C_1. Pour le cas C_2 le nombre minimum des termes est égal à 4. Il existe une infinité de formules de cette espèce dépendant de six paramètres arbitraires. *S. C. van Veen* (Delft).

Vaida, Dragos. Extension du théorème d'approximation de K. Weierstrass aux fonctions hyperboliques-continues à deux variables. Com. Acad. R. P. Romîne 6 (1956), 1173–1178. (Romanian. Russian and French summaries)

The theorem referred to in the title is that of the approximation by polynomials of continuous functions $f(x)$, defined on the interval $0\leq x\leq 1$. In the terminology of M. Nicolescu [Acad. R. P. Romîne. Stud. Cerc. Mat. 3 (1952), 7–51; MR 16, 576] the author's theorem is as follows: If $f(x, y)$ is defined on the square $0\leq x\leq 1$, $0\leq y\leq 1$, and is hyperbolically continuous, then to each $\varepsilon>0$ there corresponds a generalized pseudo-hyperbolic polynomial $h_\varepsilon(x, y)$ such that $|f(x, y)-h_\varepsilon(x, y)|\leq\varepsilon$ on the given square. The method of proof is an adaptation of S. Bernstein's [Soobsc. Har'kov. Mat. Obsc. (2) 13 (1912), 1–2] proof of Weierstrass' theorem, the essential modification arising from the fact that here, unlike in the one variable case, boundedness in absolute value of the function f is not implied by its continuity. *J. B. Diaz.*

Timan, A. F. Generalization of a theorem of Stone. Dokl. Akad. Nauk SSSR (N.S.) 111 (1956), 955–958. (Russian)

The following simple variant of the Stone-Weierstrass approximation theorem [M. H. Stone, Trans. Amer. Math. Soc. 41 (1937), 375–481] is proved. Let G be a regular topological space. For a real-valued function f on G and a real number a, let $G^a(f)=\{x: x\in G, f(x)\geq a\}$ and let $G_a(f)=\{x:x\in G, f(x)\leq a\}$. Let $\mathfrak{M}$ be the set of all bounded real-valued continuous functions on G such that, for all except possibly one real number a, at least one of the sets $G^a(f)$ or $G_a(f)$ is bicompact (the void set being taken as bicompact). Let D be a subset of $\mathfrak{M}$ that is a ring, contains all constant functions, and separates points of G. Then D is uniformly dense in $\mathfrak{M}$. *E. Hewitt.*

van der Corput, J. G. Asymptotic developments. I. Fundamental theorems of asymptotics. J. Analyse Math. 4 (1955/56), 341–418.

The present paper is a revision of "Asymptotic expansions, I, Fundamental theorems of asymptotics" [Dept. of Math., Univ. of California, Berkeley, 1954; MR 16, 352]. The Sections 7 and 8 have been completely rewritten and in Section 4 many additions have been made. Section 7 (Analytic functions with prescribed asymptotic expansion) culminates in theorem 7.4, in which the asymptotic expansions are prescribed in the neighbourhood of n different given points $\beta_1, \beta_2, \cdots, \beta_n$ of a given region S lying in the complex z-plane. Assuming that $a_{hk}(z)$ $(1\leq k\leq n;\ h\geq 0)$ is analytic in S and also in the points $\beta_1, \cdots, \beta_{k-1}, \beta_{k+1}, \cdots, \beta_n$, that $a_{hk}(z)$ $(1\leq k\leq n)$ is bounded in S in the neighbourhood of β_k and tends in S in the neighbourhood of β_k asymptotically to zero as $h\to\infty$, it is shown that it is possible to construct in S an analytic function which possesses in S in the neighbourhood of β_k $(k=1, \cdots, n)$ the prescribed asymptotic expansion $\sum'^{\infty}_{h=0} a_{hk}(z)$. In Section 8 (Equations) it is assumed that the numbers y_r $(r=1, \cdots, n)$, which may depend on ω satisfy n asymptotic equations of the form

$$\sum_{H,K}{}' a_{H\nu K}X^H Y^K \sim 0 \quad (\nu=1, 2, \cdots, n).$$

The sum is extended over the systems $H=(h_1, \cdots, h_m)$ formed by m integers ≥ 0 and over the systems $K=(k_1, \cdots, k_n)$ formed by n integers ≥ 0 with the property $|H|+|K|\geq 0$. Here X^H means $x_1^{h_1}\cdots x_m^{h_m}$, Y^K means $y_1^{k_1}\cdots y_n^{k_n}$; $|H|, |K|$ are notations for $h_1+\cdots+h_m$ and $k_1+\cdots+k$. It is shown that under general conditions these n equations determine completely the asymptotic behaviour of the numbers $y_1, \cdots, y_n$. At the end of Section 4 (Asymptotically Convergent Series) the author draws the attention to the weak point of his method. Under certain circumstances it yields for certain numbers or functions asymptotic expansions, but the notion of asymptotic convergence never can yield a numerical upper bound for the absolute value of the remainder term; it gives only the order of magnitude of this upper bound. Each method which leads to a numerical remainder bound lies outside this calculus. This fact is illustrated by an example treated by a method of Robert L. Evans [Quart. Appl. Math. 12 (1954), 295–300; MR 16, 131]. *S. C. van Veen* (Delft).

Timan, A. F.; and Tučinskiĭ, L. I. Approximation of differentiable functions given on a finite segment by algebraic polynomials. Dokl. Akad. Nauk SSSR (N.S.) 111 (1956), 771–772. (Russian)

Soit $W^rH^\alpha M$ la classe des fonctions f r-fois dérivables sur $[-1, +1]$, avec $|f^{(r)}(x_1)-f^{(r)}(x_2)|\leq M|x_1-x_2|^\alpha$; soit $S_n(f; x)$ la $n$ième somme partielle de la série de Fourier-Tchebytchev de f; soit $\mathcal{E}_n=\sup|f(x)-S_n(f; x)|$ pour $f\in W^rH^\alpha M$ et $|x|\leq 1$. Les auteurs indiquent que

$$\mathcal{E}_n= \pi^{-2}2^{\alpha+1}M \log n\cdot n^{-r-\alpha}(1-x^2)^{\frac{1}{2}(r+\alpha)}\int_0^{\frac{1}{2}\pi} t^\alpha \sin t dt+O(n^{-r-\alpha})$$

[pour $r=0$, le résultat est du à Timan, mêmes Dokl. (N.S.) 77 (1951), 969–972; MR 12, 823]. *J. P. Kahane.*

Talalyan, A. A. On convergence of orthogonal series. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 515–516. (Russian)

Let $\{\varphi_n(x)\}$ be an ortho-normal system on $0\leq x\leq 1$. A non-decreasing function $\omega(x)$, $x\geq 1$, is said to be a Weyl factor, if for any real sequence $\{a_n\}$ the condition

$$\sum a_n^2\omega(n)<\infty$$

implies that $\sum a_n\varphi_n$ converges almost everywhere; $\omega(x)$ is said to be a precise Weyl factor if for each positive $w(x)=o\{\omega(x)\}$ there is a sequence $\{a_n\}$ such that $\sum a_n^2w(n)<\infty$ and $\sum a_n\varphi_n(x)$ diverges almost everywhere. The very well-known results of Rademacher [Math. Ann. **87** (1922), 112–138] and Menchov [Fund. Math. **4** (1923), 82–105] assert that $\omega(x)=\log^2 x$ is a Weyl factor for each ortho-normal system, and is a precise Weyl factor for some orthonormal systems. In the present note the author states without proof the following results. 1) Let $\omega(x)$, $x\geq 1$, tend monotonically to $+\infty$ and suppose that for each $x\geq 1$ and $a>0$ we have

$$\omega(x+a)-\omega(x)\leq\log^2(x+a)-\log^2 x;$$

then there is an ortho-normal system $\{\varphi_n\}$ on (0, 1) for which $\omega(x)$ is a precise Weyl factor. 2) Given any $f\in L^2(0,1)$, $f\neq 0$, there is a complete ortho-normal system $\{\varphi_n\}$ on (0, 1) such that the Fourier series $\sum a_n\varphi_n$ of f diverges almost everywhere. 3) If $\{\varphi_n\}$ is a complete ortho-normal system in (0, 1), then for any measurable $f(x)$ there is a series $\sum a_n\varphi_n(x)$ converging in measure to $f(x)$ and, moreover, $\lim a_n=0$. 4) If $\{\varphi_n\}$ is the same as in 3), then there is a series $\sum a_n\varphi_n$ converging in measure to 0, with not all a_n vanishing and $a_n\to 0$. *A. Zygmund.*

Griseri, Bruna. Semplificazioni nella determinazione di alcune costanti della teoria dei polinomi ortogonali classici. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **90** (1955–56), 359–361.

General relations between the coefficients of orthogonal polynomials $P_n(x)$ satisfying the generalized Rodriguez relation and the coefficient of corresponding differential equations of $P_n(x)$ are used to express some of these numbers in terms of the others. *E. Kogbetliantz.*

Inozemcev, O. I. On the theory of best approximation of functions of several variables by means of entire functions of finite degree. Ukrain. Mat. Ž. **8** (1956), 396–412. (Russian)

The author proves the theorems announced in Dokl. Akad. Nauk SSSR (N.S.) **91** (1953), 15–18; MR **15**, 107. *R. P. Boas, Jr.* (Evanston, Ill.).

Singer, Ivan. Caractérisation des éléments de meilleure approximation dans un espace de Banach quelconque. Acta Sci. Math. Szeged **17** (1956), 181–189.

Some theorems about best approximations, including a generalization due to Remez of a Tchebychef-Bernstein theorem, are deduced from the following result: An element q of a subspace G of a Banach space E minimizes the distance from G to a given x if and only if there exists a linear functional f of norm 1 that vanishes on G and such that $f(x)=\|x-q\|$. If G is n-dimensional, then f may be found as a convex combination of at most $n+1$ extreme points of the unit ball in E^*. *M. Jerison.*

Izumi, Shin-ichi. Fourier series. IV. Korevaar's conjecture. Proc. Japan Acad. **32** (1956), 655–657.

Let f have period 2π and finite total variation V on $[-\pi,\pi]$. The method used by the reviewer [Nederl. Akad. Wetensch. Proc. Ser. A. **56** (1953), 281–293; MR **15**, 119] shows that there exists an absolute constant A such that for every n there is a trigonometric polynomial t_n of order $\leq n$ which satisfies

$$(*)\qquad \int_{-\pi}^{\pi}|f(x)-t_n(x)|dx\leq AV/n.$$

By a result of Zygmund on best uniform approximation [Duke Math. J. **12** (1945), 47–76; MR **7**, 60] the left hand side of (*) cannot be $o(1/n)$ when f is discontinuous. The method of the present paper shows that for a discontinuous f there actually is a constant $B>0$ such that for every $n\geq 1$ and every t_n the left hand side of (*) exceeds B/n. *J. Korevaar* (Madison, Wis.).

See also: Gončar, p. 886; Al-Salam, p. 886; Pogodičeva and Timan, p. 891; Luke, p. 937; Sharma, p. 963 Chandrasekhar, p. 969.

Trigonometric Series and Integrals

★ **Salem, R.; and Zygmund, A. A note on random trigonometric polynomials.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. II, pp. 243–246. University of California Press, Berkeley and Los Angeles, 1956. \$6.50.

Consider the Rademacher functions

$$\varphi_n(t)=\mathrm{sign}(\sin 2^n\pi t)\quad(n\geq 0);$$

choose $t\in[0,1]$ with the uniform distribution

$$P(t\in ds)=ds,$$

and form the trigonometric polynomial I_n of degree n with $I_n(2\pi m(2n+1)^{-1})=\varphi_m(t)$ $(0\leq m\leq 2n)$. Continuing their joint work on trigonometric sums with random signs [Acta Math. **91** (1954), 245–301; MR **16**, 467], the authors show that

$$P(\limsup_{n\uparrow+\infty}(\log n)^{-\frac{1}{2}}\|I_n\|_\infty\leq 2)=1.$$

H. P. McKean, Jr. (Princeton, N.J.).

Pogodičeva, N. A.; and Timan, A. F. On a certain relation in the theory of summation of interpolation polynomials and Fourier series. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 542–543. (Russian)

Let $f(x)$ be continuous of period 2π, and let

$$S_n(f;x)=\tfrac{1}{2}a_0+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx),$$

$$\tilde{S}_n^{(n)}(f;x)=\tfrac{1}{2}a_0^{(n)}+\sum_{k=1}^{n}(a_k^{(n)}\cos kx+b_k^{(n)}\sin kx)$$

be respectively the nth partial sum of the Fourier series of f, and the Lagrange interpolating polynomial of order n coinciding with f at the equidistant fundamental points $x_\nu^{(n)}=2\pi\nu/(2n+1)$ $(\nu=0,1,\cdots,2n)$. Given a triangular matrix of numbers $\lambda_k^{(n)}$ $(k=0,1,2,\cdots,n+1;\ \lambda_0^{(n)}=1,\ \lambda_{n+1}^{(n)}=0)$ we define two procedures of approximating f by trigonometric polynomials by multiplying the kth terms of S_n and $\tilde{S}_n^{(n)}$ by $\lambda_k^{(n)}$ $(k=0,1,\cdots,n)$ and denoting the resulting expressions by $u_n(f;x;\lambda)$ and $\tilde{u}_n(f;x;\lambda)$ respectively. Let

$$L_n=\sup_{|f|\leq 1}|u_n(f;x;\lambda)|,\quad L_n(x)=\sup_{|f|\leq 1}|\tilde{u}_n(f;x;\lambda)|.$$

The following result is stated without proof. Suppose that $|\lambda_k^{(n)}|\leq M$, and that for each fixed n the sequence $\lambda_0^{(n)}=1, \lambda_1^{(n)}, \cdots, \lambda_{n+1}^{(n)}=0$ is either concave or convex. Then, as $n\to+\infty$,

$$L_n(x)=\tfrac{1}{2}\pi|\sin(n+\tfrac{1}{2})x|L_n+O(1),$$

where $O(1)$ is a quantity uniformly bounded in x and n by a constant which depends on M only. *A. Zygmund.*

Matsumoto, Kishi. On absolute Cesàro summability of a series related to a Fourier series. Tôhoku Math. J. (2) 8 (1956), 205–222.

Ist $\sum_{n=0}^{\infty} A_n(t)$ die Fourierreihe einer Funktion

$$f(t)\in L(-\pi,+\pi)$$

und ist

$$\varphi_\alpha(t)=\Gamma(\alpha+1)t^{-\alpha}\varphi_\alpha(t),$$

$$\varphi_\alpha(t)=(\Gamma(\alpha))^{-1}\int_0^t(t-u)^{\alpha-1}\varphi(u)du \quad (\alpha>0)$$

mit $\varphi(t)=\varphi_0(t)=\frac{1}{2}\{f(x+t)+f(x-t)\}$, so gilt: 1) Ist

$$\int_0^\pi t^{-\gamma+\beta}|d\varphi_\beta(t)|<\infty \quad (1>\alpha>\gamma\geq\beta\geq0),$$

so ist $\sum n^{\gamma-\alpha}A_n(t)$ $|C,\alpha|$-summierbar für $t=x$. 2) Ist

$$\int_0^\pi |dt^{-\gamma}\varphi_\beta(t)|<\infty \quad (1\geq\gamma\geq\beta\geq0),$$

so ist $\sum n^{\gamma-\beta}A_n(t)/(\log(n+2))^{1+\varepsilon}$ $|C,\gamma|$-summierbar für $t=x$ und für jedes $\varepsilon>0$. Bei den Beweisen wird ganz direkt vorgegangen. Die Glieder der Cesàro-Transformation der betrachteten Reihen sind Integraltransformationen der Funktionen $\varphi_\alpha(t)$; die auftretenden Kerne werden abgeschätzt. Bekannte Spezialfälle bei 1) sind $\beta=\gamma$ [Bosanquet, Proc. London Math. Soc. (2) **41** (1936), 517–528] und $\beta=0$ [Mohanty, J. London Math. Soc. **25** (1950), 63–67; MR **11**, 592] und bei 2) $\beta=\gamma$ [Cheng, Duke Math. J. **15** (1948), 29–36; MR **9**, 580, 735]. Unter der Voraussetzung von 2) kann für kein $\alpha>0$ auf die $|C,\alpha|$-Summierbarkeit von $\sum n^{\gamma-\beta}A_n(t)$ geschlossen werden.

A. Peyerimhoff (Giessen).

Kanno, Kôsi. On the Riesz summability of Fourier series. Tôhoku Math. J. (2) 8 (1956), 223–234.

Ist $\varphi(t)\in L(-\pi,+\pi)$ eine gerade Funktion und ist für $\alpha>0$ $\varphi_\alpha(t)=(\Gamma(\alpha))^{-1}\int_0^t \varphi(u)(t-u)^{\alpha-1}du$, so folgt aus

$$\varphi_\alpha(t)=o(t^\alpha/(\log t^{-1})^{\alpha\beta}) \quad (t\to\infty)$$

für $\beta>0$ und jedes $\eta>\alpha$ (α ganz) bzw. $\eta>[\alpha]+1$ (α nicht ganz) die $(R, \exp(\log\omega)^{1+\beta}, \eta)$-Summierbarkeit der Fourierreihe von $\varphi(t)$ für $t=0$. Dieser Satz enthält Ergebnisse von F. T. Wang [J. London Math. Soc. **18** (1943), 155–160; **5**, 237; Proc. London Math. Soc. (2) **51** (1949), 215–231; MR **5**, 237; **11**, 27].

A. Peyerimhoff (Giessen).

Prasad, B. N.; and Bhatt, S. N. The summability factors of a Fourier series. Duke Math. J. 24 (1957), 103–117.

Die Arbeit befasst sich mit der Frage nach der absoluten Cesàro-Summierbarkeit von Reihen $\sum\lambda_n c_n(t)$ mit

$$f(t)\sim\sum_1^\infty c_n(t),$$

$\{\lambda_n\}$ konvex und $\sum\lambda_n/n$ konvergiert. Die hier bekannten Ergebnisse werden zunächst ausführlich diskutiert; anschliessend werden einige Verschärfungen angegeben: 1) Ist

$$\int_0^t\varphi^*(u)du=O\left\{t\left(\log\frac{1}{t}\right)^\beta\right\} \quad (\beta\geq0;\ t\to0)$$

$$(\varphi^*(u)=\tfrac{1}{2}\{f(x+u)+f(x-u)-2f(x)\})$$

so ist $\sum\lambda_n c_n(x)/(\log(n+1))^{1+\beta}$ $|C,1|$-summierbar. 2) Ist

$$\int_t^\delta\left|\frac{\varphi^*(u)}{u}\right|du=O\left(\left(\log\frac{1}{t}\right)^k\right) \quad (k>0;\ t\to0)$$

füt ein $0<\delta<\pi$, so ist $\sum\lambda_n c_n(x)/(\log(n+1))^k$ $|C,1|$-summierbar. 3) Ist $\varphi_\alpha(t)(\varphi_\alpha(t)=\alpha t^{-\alpha}\int_0^t(t-u)^{\alpha-1}\varphi(u)du$ $(\alpha>0)$, $\varphi(u)=\varphi_0(u)=\frac{1}{2}\{f(x+u)+f(x-u)\})$ für ein $0\leq\alpha\leq1$ von beschränkter Schwankung in $(0,\pi)$, so ist $\sum\lambda_n c_n(x)$ $|C,\alpha|$-summierbar. 4) Ist

$$\int_0^t u|d\varphi_\alpha(u)|=O(t) \quad (0<\alpha\leq1;\ 0\leq t\leq\pi),$$

so ist $\sum\lambda_n c_n(x)$ $|C,\beta|$- $(\beta>\alpha)$ und $\sum\lambda_n c_n(x)/\log(n+1)$ $|C,\alpha|$-summierbar. Die Beweise beruhen wesentlich auf dem folgenden reihentheoretischen Satz: Aus

$$\theta_n^\alpha=O((\log n)^k) \quad (C,1),$$

wobei $k\geq0$ (mit $\theta_n^\alpha=|t_n^\alpha|$ $(\alpha=1)$, $\theta_n^\alpha=\max_{1\leq\nu\leq n}|t_\nu^\alpha|$ $(0<\alpha<1)$; $t_n^\alpha=(A_n^\alpha)^{-1}\sum_{\nu=1}^n A_{n-\nu}^{\alpha-1}\nu a_\nu$, $A_n^\alpha=\binom{n+\alpha}{n}$), folgt, dass $\sum\lambda_n a_n/(\log(n+1))^k$ $|C,\alpha|$-summierbar ist.

A. Peyerimhoff (Giessen).

Ul'yanov, P. L. The A-integral and conjugate functions. Moskov. Gos. Univ. Uč. Zap. 181. Mat. 8 (1956), 139–157. (Russian)

The author proves the theorems announced in Uspehi Mat. Nauk (N.S.) **10** (1955), no. 1(63), 189–191.

R. P. Boas, Jr. (Evanston, Ill.).

Korenblyum, B. I. Harmonic analysis of fast growing functions. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 201–203. (Russian)

Let L be the space of functions on $(-\infty,\infty)$ which satisfy $\|f\|=\int|f(x)|\exp\alpha|x|dx<\infty$ $(\alpha>0$, fixed). The space M of continuous linear functionals may be identified with the measurable functions which satisfy

$$\|g\|=\text{ess sup}\{|g(x)|\exp(-\alpha|x|)\}<\infty.$$

Let $I(f)$ be the ideal generated by $f\in L$ and

$$F(z)=\int f(x)\exp(-ixz)dx \quad (|Im\ z|\leq\alpha).$$

Theorem 1. $I(f)=L$ if and only if (1) $F(z)\neq0$, and

$$(2)\qquad \begin{aligned}\gamma_1&=\lim\sup_{x\to+\infty}\ln|F(x)|/\exp(\pi x/2a)=0,\\ \gamma_2&=\lim\sup_{x\to-\infty}\ln|F(x)|/\exp(-\pi x/2a)=0.\end{aligned}$$

If (1) is fulfilled but one of γ_1 or γ_2 is negative, $I(f)$ is the set of all f^* for which the corresponding numbers γ_1^*, γ_2^* satisfy $\gamma_1^*\leq\gamma_1$, $\gamma_2^*\leq\gamma_2$.

Theorem 2. Let $0\neq g\in M$ and $I(g)$ the smallest weakly closed linear manifold containing g and invariant under translations. $I(g)$ contains at least one of the functions, $\exp(-i\lambda x)$ $(|\text{Im}\ \lambda|\leq\alpha)$, $\exp(-i\mu_1 x)/\Gamma(1+2\alpha xi/\pi)$ or $\exp(-i\mu_2 x)/\Gamma(1-2\alpha xi/\pi)$ (μ_1, μ_2 real). The sets of numbers $\{\lambda\}$, $\{\mu_1\}$, $\{\mu_2\}$ of theorem 2 are called the harmonic, right non-harmonic and left non-harmonic spectrum of g respectively.

Theorem 3. If g has no non-harmonic spectrum and its harmonic spectrum is bounded, then $g(x)$ coincides a.e. on the reals with an entire function of exponential type.

No proofs are given. *A. Devinatz* (St. Louis, Mo.).

Duffin, R. J. Representation of Fourier integrals as sums. III. Proc. Amer. Math. Soc. 8 (1957), 272–277.

[For parts I and II see Bull. Amer. Math. Soc. 51 (1945), 447–455; Proc. Amer. Math. Soc. 1 (1950), 250–255; MR 6, 266; 11, 592; 12, 1002]. The author is concerned with the functions

$$f(x)=\sum_1^\infty n^{-1}\alpha_n\phi(x/n),\ g(x)=\sum_1^\infty x^{-1}\alpha_n\phi(n/x),$$

where ϕ is "arbitrary," $\alpha_n=\sin\frac{1}{2}\pi n$; they are formally a transform pair with kernel $\sin\frac{1}{2}\pi x$. By formal application of the Möbius inversion we have

$$(*)\qquad f(x)=\sum_1^\infty (mx)^{-1}\beta_m\sum_1^\infty \alpha_n g(n/(mx))$$

[cf. Wintner, Amer. J. Math. 69 (1947), 685–708; MR 9, 279; Goldberg and Varga, Duke Math. J. 23 (1956), 553–559; MR 18, 304]. The aim of the present paper is to validate (*) for two sets of hypotheses on ϕ. The author finds it sufficient to suppose that $\alpha_n\alpha_m=\alpha_{n+m}$, α_n bounded, $\alpha_1=1$; let $\beta_n=\mu_n\alpha_n$, where μ_n is the Möbius symbol. He says that $\sum c_n$ is P-summable if $\sum_{n=1}^\infty \delta_n{}^i c_n$ approaches a limit as $i\to\infty$, where $\delta_n{}^i=1$ unless n contains a prime factor greater than the ith prime, otherwise $\delta_n{}^i=0$. (1) If $(1+x^{-1})\phi(x)\in L(0,\infty)$, the series defining f and g converge almost everywhere and

$$\sum\{\beta_n/(nx)\}g(1/(nx))=\sum(\beta_n/n)f(x/n),$$

the series being P-summable. (2) Let $f\in L$ and let g be its Fourier cosine transform with kernel $2\cos 2\pi x$. Then

$$\sum_{m=1}^\infty(\mu_m/(mx))\sum_{n=1}^\infty e^{-2\pi yn(m_x)}g(n/(mx))$$

is, for positive x and y, equal to the (half-plane) Poisson integral of $f(x)$, the outer sum being P-summable. (3) Let f be normalized and of bounded variation on $(0,\infty)$ with $f(\infty)=0$. Then for positive x,

$$\sum_{m=1}^\infty(-\mu_{2m})/(2mx)\sum_{n=1}^\infty(-1)^n f(n/(2mx))=2\int_0^\infty f(t)\cos 2\pi xt\,dt,$$

where the outer sum converges uniformly if multiplied by x. *R. P. Boas, Jr.* (Evanston). Ill.).

See also: Geronimus, p. 879; Bredihina, p. 886; Kupcov, p. 886; Sunouchi, p. 889; Izumi, p. 891; Bochner, p. 940; Pitman, p. 955; Lyapin, p. 978.

Integral Transforms

Sears, D. B. Integral transforms over certain function spaces. I. Quart. J. Math. Oxford Ser. (2) 8 (1957), 68–80.

Let (δ, s_0, μ) and (Δ, S_0, ν) be measure spaces, i.e., δ and Δ are sets, s_0 and S_0 are σ-rings (in the sense of Halmos) and μ and ν are measures. Let $s\subseteq s_0$ and $S\subseteq S_0$ be subrings for which every element is of finite measure and such that the linear manifold determined by the characteristic functions of elements of s and S are dense in l^2 and L^2, where l^2 and L^2 are the spaces of square integrable functions over (δ, s_0, μ) and (Δ, S_0, ν) respectively. Let $\gamma(P,x)$ and $\Gamma(p,y)$ be defined on $S\times\delta$ and $s\times\Delta$ respectively. Necessary and sufficient conditions are given, in terms of relations between γ, Γ, μ and ν such that the transformations from l^2 to L^2 and L^2 to l^2 given by

$$\int_P F(y)d\nu(y)=\int_\delta f(x)\bar\gamma(P,x)d\mu(x),$$

$$\int_p f(x)d\mu(x)=\int_\Delta F(y)\bar\Gamma(p,y)d\nu(y)$$

respectively should be unitary and inverses of each other. These conditions are interpreted for special classes of kernels γ and Γ and spaces (δ, s_0, μ) and (Δ, S_0, ν). The results given generalize theorems of Bochner and Chandrasekharan [Fourier transforms, Princeton, 1949; MR 11, 173], and Temple [Proc. London Math. Soc. (2) 31 (1930), 231–242, 243–252]. *A. Devinatz.*

González Domínguez, A.; and Scarfiello, R. Note on the formula v.p. $1/x\cdot\delta=-\frac{1}{2}\delta'$. Rev. Un. Mat. Argentina 17 (1955), 53–67 (1956). (Spanish)

Let $g_n(x)$ be a singular kernel which satisfies the conditions: a) $\lim_{n\to\infty}\int_{-\infty}^\infty g_n(x)dx=1$; b) $\int_{-\infty}^\infty|g_n(x)|dx<M$; c) $\lim_{n\to\infty}\int_I g_n(x)dx=0$; for every interval I which excludes the origin. Assume that the derivative of $g_n(x)$ is bounded for every n and that d) $|xh_n(x)|<N$, where $h_n(x)=\mathrm{P}\int_{-\infty}^\infty g_n(y)dy/(x-y)$, P means the principal value of the integral. Finally let $K_n(x)=g_n(x)h_n(x)$. The following propositions are proved:

Lemma 1. $\lim_{n\to\infty}\int_{-\infty}^\infty xK_n(x)dx=1$; $\lim_{n\to\infty}\int_{-\infty}^\infty K_n(x)dx=0.$

Theorem 1.

$$\lim_{n\to\infty}\int_{-\infty}^\infty xK_n(x)f(x)dx=\tfrac{1}{2}f(0)$$

for every bounded function continuous at the origin. Theorem 2.

$$\lim_{n\to\infty}\int_{-\infty}^\infty K_n(x)f(x)dx=\tfrac{1}{2}f'(0)$$

for every bounded function with a continuous derivative at the origin.

Examples are the Fejer, Poisson and Weierstrass kernels $g_n(x)$. For the Dirichlet kernel theorems 1 and 2 only hold under more restrictive conditions on $f(x)$.

Theorem 2 can be written as a limit in the sense of Schwartz:

$$\lim_{n\to\infty}g_n(x)\Big(g_n(x)*\mathrm{P}\frac{1}{x}\Big)=-\frac{1}{2}\delta'(x),$$

where $*$ means the convolution, or symbolically as: $\delta(x)\mathrm{P}\,1/x=-\frac{1}{2}\delta'(x)$ a formula which occurs in electrodynamics [cf. W. Heitler, Quantum theory of radiation, 3rd ed., Oxford, 1954, p. 309] and is usually justified by consideration of the Poisson representation of $\delta(x)$ and P/x. *J. Leite Lopes* (Pasadena, Calif.).

Cotlar, Mischa. A unified theory of Hilbert transforms and ergodic theorems. Rev. Mat. Cuyana 1 (1955), 105–167 (1956). (Spanish summary)

A general theory is expounded which contains, as special cases, the Hilbert transform theorems (Lusin-Riesz-Kolmogoroff one-dimensional Hilbert transform theorem and Zygmund-Calderon higher dimensional Hilbert transform theorem) and the ergodic theorems (von Neumann mean ergodic theorem, G. D. Birkhoff individual ergodic theorem and N. Wiener dominated ergodic theorem). Let $R^n=\{x\}$ be the n-dimensional euclidean space, and let $\Omega=\{P\}$ be an abstract space

endowed with a finite total measure and an n-dimensional continuous group $\{\sigma_x\}$ of equi-measure transformations in Ω. Define the operators

$$H_m f(P)=\int_{R^n} f(\sigma_x P)K^{(m)}(x)dx.$$

If $\Omega=R^n$, $\sigma_x t=x+t$, and $K^{(m)}(x)=\sum_{i=-m}^{m}2^{-ni}K(2^{-i}x)$, where $K(x)=\omega(x)/|x|^n$ for $1\leq|x|\leq 2$ and zero otherwise, then, under an appropriate assumption upon $\omega(x)$, $Hf=\lim_{m\to\infty}H_m f$ is the n-dimensional Hilbert transform. If $K(x)=-1$ for $|x|<1$, $K(x)=1$ for $1\leq|x|\leq 2$ and zero otherwise, then

$$H_m f(P)=2^{-m+1}\int_{R^n} f(\sigma_x P)dx-(1+2^{-1}+\cdots+2^{-m+1})\times\int_{|x|\leq 1} f(\sigma_x P)dx$$

is the n-dimensional ergodic operator. It is proved that, for $f\in L^p(\Omega)$ with $1<p<\infty$, $H_m f(P)$ converges in the $L^p(\Omega)$-mean as well as almost everywhere to $Hf(P)$, being dominated by an $L^p(\Omega)$ function. For the proof, the Saks theorem plays a fundamental role to ensure continuity, at $0\in L^p(\Omega)$, of the operator $\tilde{H}f(P)=\sup_m H_m f(P)-\inf_m H_m f(P)$ from $L^p(\Omega)$ to the Frechet space $S(\Omega)$. [Cf. the reviewer's deduction of the individual ergodic theorem from the mean ergodic theorem, Jap. J. Math. 17 (1940), 31–36; MR 2, 105.] *K. Yosida* (Tokyo).

Calderón, A. P.; and Zygmund, A. On singular integrals. Amer. J. Math. 78 (1956), 289–309.

As in a previous paper [Acta Math. 88 (1952), 85–139; MR 14, 637] the authors deal with extensions of Hilbert transforms to the n-dimensional space E_n. Let $x, y, z, \cdots$ be vectors in E_n, $|x|$ the length of x, and

$$f_\varepsilon(x)=\int_{|x-y|>\varepsilon} K(x, y)f(y)dy,$$

where dy is the element of volume in E_n. The method which the authors use to deduce properties of f_ε is entirely different from their previous procedure. It is based on the classical theory of Hilbert transforms in E_1 (where the kernel $K(x, y)$ is $\pi^{-1}(x-y)^{-1}$); and it yields, except for some cases including $f\in L$, their previous results under far less restrictive assumptions.

In Theorem 1, the kernel is $K(x, y)=N(x-y)$, where $N(x)$ is homogeneous of degree $-n$; in Theorem 2, $K(x, y)=N(x, x-y)$, where $N(x, y)$ is homogeneous in y; in Theorems 3 and 4, $K=N(x, x-y)\psi(|x-y|)$, where $\psi(t)$ is an even or odd Fourier-Stieltjes transform, while $N(x, y)$ is odd or even, respectively, and homogeneous in y and its modulus is majorised by a homogeneous function $F(y)$. In each case, $f(x)\in L^p$, and N satisfies some conditions of integrability. Then, as in the classical case, $f_\varepsilon(x)$ tends to a function $\tilde{f}(x)$ both in the mean of order p, and pointwise almost everywhere, as $\varepsilon\to 0$, provided that p exceeds 1 and is subject to some further restrictions in Theorems 2 and 4. Again

$$\int_{-\infty}^{\infty}(\sup_\varepsilon|f_\varepsilon(x)|)^p dx\leq A^p\int_{-\infty}^{\infty}|f(x)|^p dx,$$

where A is independent of $f(x)$. Two other kernels are dealt with in Theorems 6 and 7, while in Theorem 5 the convergence to f, in the mean of order p, of the spherical means of order $\frac{1}{2}n(n-1)$ of the Fourier integral representation of $f\in L^p$ (n variables, n odd; $1<p\leq 2$) is stated, which is proved at the end of the paper.

First the Theorems 3, 4, 6 and 7 are deduced from the classical case an by interesting technique; while the proofs of 1 and 2 are more complicated: they depend on vector valued functions and are based on theorems 3 and 4 and on the transforms

$$\int_{|x-y|>\varepsilon} R(x-y)F(y)dy,\quad \int_{|x-y|>\varepsilon} R(x-y)F(y)\phi(|y|)dy,$$

where $R(x)$ is the (vector valued and odd) Riesz kernel $R(x)=\pi^{-\frac{1}{2}(n+1)}\Gamma(\frac{1}{2}n+\frac{1}{2})x|x|^{-n-1}$, $F(x)$ is even, homogeneous and satisfies some condition of integrability, and where $\phi(t)$ $(0\leq t<\infty)$ is some continuously differentiable function. *H. Kober* (Birmingham).

Calderon, A. P.; and Zygmund, A. Addenda to the paper "On a problem of Mihlin". Trans. Amer. Math. Soc. 84 (1957), 559–560.

Corrigenda to same Trans. 78 (1955), 209–224 [MR 16, 816]. Only the proofs are affected; the results remain as before. *F. Smithies* (Cambridge, England).

★**Doetsch, Gustav. Handbuch der Laplace-Transformation. Band III. Anwendungen der Laplace-Transformation, 2. Abteilung.** Birkhäuser Verlag, Basel und Stuttgart, 1956. 300 pp. 40.00 francs suisses.

The present volume brings to a conclusion a three volume treatise on the Laplace transform [for reviews of vols. 1 (1950) and 2 (1955) see MR 13, 230; 18, 35]. The parts and chapters of the present volume are numbered consecutively with those of volume 2. The volume opens with part 4 (parts 1, 2, and 3 comprising volume 2) which is devoted to the theory of partial differential equations. Topics covered include equations with constant coefficients, equations with variable coefficients, uniqueness and compatibility conditions, and the principles of Huygens and Euler. Many partial differential equations familiar in diverse applied fields make their appearance here as examples. Part 5 is concerned with difference equations, both ordinary and partial. A brief treatment is also accorded to mixed differential-difference equations. In part 6 integral equations are taken up. The principal role is accorded to those of convolution or of Wiener-Hopf type, although other types make their appearance as well. Application is made to the transcendental addition formulas of various special functions. Finally a rather detailed study is made of complex convolutions. Part 7 continues the subject of finite Laplace transforms and entire functions of exponential type, a subject already introduced in volume 1.

All three volumes share a common bibliography which is placed at the end of volume 1. (A short supplement occurs at the end of the present volume.)

This volume like those which preceded it is written in a very careful, clear, and detailed manner. Very little specialized knowledge beyond a basic familiarity with the elements of real and complex analysis is assumed. Because of this the usefulness of this work to the non-specialist (as well as to the specialist) should be very great. *I. I. Hirschman* (St. Louis, Mo.).

Gilbert, Walter M. Completely monotonic functions on cones. Pacific J. Math. 6 (1956), 685–689.

The Hausdorff-Bernstein-Widder theorem states that completely monotonic functions are the class of Laplace-Stieltjes transforms of bounded monotone functions. The definition of completely monotonic and the theorem have been extended from the positive real axis in one dimension

to the positive orthant, $x_i \geqq 0$ in n dimensions. In the present note an extension is made to n-dimensional cones with vertex at the origin. And a result on the convex closure of a set of overlapping cones on each of which the function is completely monotonic is obtained.

P. Franklin (Cambridge, Mass.).

McCrossen, Garner. A generalized Laplace-Stieltjes transformation. Proc. Amer. Math. Soc. 8 (1957), 278–285.

Cesàro means are applied to the Laplace-Stieltjes transform. Set

$$G(t, s) = \int_0^t e^{-su} dF(u)$$

and let

$$f_k(s) = \lim_{t\to\infty} \frac{k}{t^k} \{G(t, s) * t^{k-1}\}$$

be the Cesàro mean of order k of $G(t, s)$.

It is shown that if $f_k(s_0)$ exists then $f_k(s)$ exists for $R(s) > R(s_0)$, and several expressions for $f_k(s)$ in terms of $G(t, s_0)$ are obtained. Some properties of $f_k(s)$ are also derived. The results and methods of proof parallel those in Ch. 9 of vol. I of Doetsch, Handbuch der Laplace-Transformation [Birkhäuser, Basel, 1950; MR 13, 230].

F. Goodspeed (Vancouver, B.C.).

Griffith, J. L. A note on two dimensional Fourier transforms. J. Proc. Roy. Soc. New South Wales 90 (1956), 134–139 (1957).

Let $\bar{w}(\xi, \eta) = f[w(x, y)]$ where

$$f[w(x, y)] = (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\xi x + \eta y)} w(x, y)\, dx dy$$

and let $\rho^2 = \xi^2 + \eta^2$. It is shown that under suitable assumptions

$$f^{-1}\left[\frac{e^{-a\rho}}{\rho} \cdot \bar{w}(\xi, \eta)\right] = -\frac{1}{2\pi} \int_{\infty}^{o(a)} \int_0^{2\pi} \frac{z}{(z^2+a^2)^{\frac12}} w(x - z\cos\theta, y - z\sin\theta)\, d\theta dz.$$

Here $\int_\infty^{o(a)} dz$ starts at $+\infty$ on the real axis and terminates at $z=0$ with $\arg z = \arg a$. Several other similar formulas are given.

I. I. Hirschman, Jr. (St. Louis, Mo.).

Akutowicz, Edwin J. On the determination of the phase of a Fourier integral. II. Proc. Amer. Math. Soc. 8 (1957), 234–238.

In part I [same Trans. 83 (1956), 179–192; MR 18, 304] the author investigated the extent to which the modulus of the Fourier transform of a function $\phi(t)$ determines $\phi(t)$ when ϕ belongs to L and L^2, and vanishes on a half-line. Here he considers how much the indetermination of ϕ is reduced when ϕ vanishes outside some finite interval. The effect of the additional assumption is to reduce the number of zeros for the Blaschke products which produce the essential indetermination by appearing as multipliers of the Fourier transform of ϕ.

R. P. Boas, Jr. (Evanston, Ill.).

Lyttkens, Sonja. The remainder in Tauberian theorems. II. Ark. Mat. 3 (1957), 315–349.

[For the part I see Ark. Mat. 2 (1954), 575–588; MR 15, 858.] The problem is to examine the implications of a Tauberian relation of the form:

$$\dot{\Phi}(x) = \int_{-\infty}^{\infty} \Phi(x-u)\, dF(u) = O\{[P(x)]^{-1}\} \text{ as } x\to\infty,$$

implies

$$\Phi(x) = O\{[P(x)]^{-\theta}\} \text{ as } x\to\infty,$$

provided $\Phi(x)$ is bounded and satisfies some growth condition. Here $F(u) \in \mathrm{BV}[-\infty, \infty]$ and $P(x)$ becomes exponentially infinite with x. In a general way, such an implication presupposes that the Fourier-Stieltjes transform $f(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} dF(x)$ is $\neq 0$ and admits of an analytic extension $f(\xi + i\eta)$ in a strip $-a < \eta < 0$, where $f(\zeta) \neq 0$ and satisfies a mild growth condition. Here

$$a = \lim x^{-1} \log P(x).$$

In one direction the author proves that if such a function $f(\zeta)$ exists and if $\int_{-\infty}^{\infty} (1+|\xi|)^{-2q} |f(\xi+i\eta)|^{-2} d\xi$ is bounded in the strip for some $q>0$, if $p(x)$ is a positive submultiplicative weight function, then $P(x) = x^{\frac12+\delta} p(x)$, $\delta > 0$, is admissible for every $\theta < 1/(q+1)$ provided

$$\Phi(x) + \int_0^x [p(u)]^{\varepsilon} du$$

is non-decreasing for every $\varepsilon > 0$ and $x > x_\varepsilon$. If $\tau > 0$ is given, if $f(\xi) \neq 0$, then a necessary and sufficient condition for the existence of a function $\Phi(x)$, bounded, continuous, $\Phi(x) \not\equiv 0$, such that $\Phi(x)$ and $e^{\tau x}\dot{\Phi}(x) \in L_2(-\infty, \infty)$ and $\dot{\Phi}(x) = O(e^{-\tau x})$, is that

$$\int_{-\infty}^{\infty} \exp(-\pi|\xi|/\tau) |\log |f(\xi)|| d\xi < \infty.$$

If $\int_{-\infty}^{\infty} |x|^{\frac12+\delta} |dF(x)| < \infty$, then the L_2-conditions may be dispensed with. In this case, the existence of a bounded $\Phi(x)$ with $\dot{\Phi}(x) = O(e^{-\tau x})$ implies the existence of the integral with π/τ replaced by any larger constant c. Moreover, under the same assumption on $F(x)$, if the Tauberian relation holds for $P(x) = e^{\gamma x}$ for every γ, $0 < \gamma < \alpha$, then $f(\zeta)$ exists and $1/f(\zeta)$ is analytic in $-\theta\alpha < \eta < 0$. In the particular case in which $1/f(\zeta)$ is analytic in the strip $-a < \eta < 0$ and its derivative is $O(1+|\xi|)^{r-1}$ in the strip, then $\theta < 1/(1+r)$ and no larger value is admissible. Finally, $\exp(-c|\xi|) \log |f(\xi)| \in L(-\infty, \infty)$ together with the Tauberian relations for $P(x) = e^{\gamma x}$, $0 < \gamma < \alpha$, implies the existence and analyticity of $1/f(\zeta)$ in a strip.

E. Hille (New Haven, Conn.).

Avetisyan, A. E. On the theory of generalized integral transforms of functions of several variables. Akad. Nauk Armyan. SSR. Izv. Fiz. Mat. Estest. Tehn. Nauki 9 (1956), no. 5, 3–24. (Russian. Armenian summary)

Džrbašyan [Izv. Akad. Nauk. SSSR. Ser. Mat. 19 (1955), 133–190; MR 16, 1102] obtained results in the theory of generalized Fourier transforms with kernel $E_\rho(zt, \mu)$ where $E_\rho(z, \mu)$ is the Mittag-Leffler function $\sum_{n=0}^{\infty} z^n/\Gamma(\mu + n/\rho)$. This paper gives direct generalizations of these results to functions of two variables with kernels of the type

$$E_{\rho_1}(z_1 t_1\, \mu_1) E_{\rho_2}(z_2 t_2, \mu_2).$$

F. Goodspeed (Vancouver, B.C.).

Heywood, Philip. Integrability theorems for power series and Laplace transforms. II. J. London Math. Soc. 32 (1957), 22–27.

Let $F(x)=\sum_{n=1}^{\infty} c_n x^n$ converge for $|x|<1$. The author shows that the following results remain true if his previous hypothesis $c_n>0$ [same J. 30 (1955), 302–310; MR 16, 1100] is replaced by $c_n>-K/n^{\gamma+\varepsilon}$ for some $\varepsilon>0$. For $\gamma<1$, and for $1\leq\gamma<2$ provided $\sum c_n=0$, the function $(1-x)^{-\gamma}F(x)$ is in $L(0,1)$ if and only if the series $\sum n^{\gamma-1}|c_n|$ converges ($\gamma\neq1$), if and only if $\sum |c_n|\log n$ converges ($\gamma=1$). Analogous results are obtained for Laplace transforms. It is pointed out that the results as stated are false if ε is omitted. *J. Korevaar* (Madison, Wis.).

Kennedy, P. B. General integrability theorems for power series. J. London Math. Soc. 32 (1957), 58–62.

The author generalizes the result on power series of Heywood's paper cited in the preceding review. Let $F(x)=\sum_{n=1}^{\infty} c_n x^n$, where $c_n\geq0$, converge for $|x|<1$. It is shown that for rather general $\varphi>0$ the product $\varphi(x)F(x)$ is in $L(0,1)$ if and only if the series $\sum c_n \int_{1-1/n}^{1}\varphi(x)dx$ converges. We quote the following corollary of this and a more refined theorem. $(1-x)^{-\gamma}\{\log 2/(1-x)\}^{-\delta}F(x)$ is in $L(0,1)$ if and only if $\sum n^{\gamma-1}(\log n)^{-\delta}c_n$ converges ($\gamma<1$), if and only if $\sum(\log n)^{1-\delta}c_n$ converges ($\gamma=1$, $\delta>1$).
J. Korevaar (Madison, Wis.).

Boas, R. P., Jr.; and González-Fernández, J. M. Integrability theorems for Laplace-Stieltjes transforms. J. London Math. Soc. 32 (1957), 48–53.

The authors' work unifies and extends the results of Heywood's first paper cited in the preceding reviews. It may be noted that Kennedy's results [see the preceding review] can also be derived from their basic theorems which are formulated for Laplace-Stieltjes transforms $\Phi(s)=\int_0^{+\infty} e^{-st}d\mu(t)$. Integrability of a rather general product $\psi(s)\Phi(s)$ over a neighborhood of $+\infty$ (of $0+$) is shown to be necessary and sufficient for integrability of a corresponding product $\Psi(t)d\mu(t)$ over a neighborhood of $0+$ (of $+\infty$). In the case where one goes from $+\infty$ to $0+$ the function Ψ is simply the Laplace transform of ψ; in the other case the relationship is somewhat more complicated.

It is informative to quote the authors' corollary for power series. Let $F(x)=\sum_{n=1}^{\infty} c_n x^n$, where $c_n\geq0$, be convergent for $|x|<1$. For $\gamma<1$, and for $k\leq\gamma<k+1$ (k a positive integer) provided

$$\sum c_n=\sum nc_n=\cdots=\sum n(n-1)\cdots(n-k)c_n=0,$$

one has the following results. (i) $(1-x)^{-\gamma}F(x)$ is in $L(0,1)$ if and only if $\sum n^{\gamma-1}c_n$ converges ($\gamma\neq k$), if and only if $\sum n^{k-1}c_n\log n$ converges ($\gamma=k$); (ii)

$$(1-x)^{-\gamma}\{\log 2/(1-x)\}^{-\delta}F(x)$$

is in $L(0,1)$ if and only if $\sum n^{\gamma-1}(\log n)^{-\delta}c_n$ converges ($\gamma\neq k$), if and only if $\sum n^{k-1}(\log n)^{1-\delta}c_n$ converges ($\gamma=k$, $\delta\neq1$), if and only if $\sum n^{k-1}(\log\log n)c_n$ converges ($\gamma=k$, $\delta=1$); etc. *J. Korevaar* (Madison, Wis.).

Lavoine, Jean. Sur le passage de l'image de $g(t)$ à celle de $g(it)$ dans la transformation de Laplace. C. R. Acad. Sci. Paris 244 (1957), 991–993.

Let g be analytic for $\operatorname{Re} z>-\alpha$, $\operatorname{Im} z>-\alpha(\alpha>0)$, regular for $\operatorname{Re} z>0$, $\operatorname{Im} z>-\alpha$. On the part of the imaginary axis where $\operatorname{Im} z>-\alpha$ the function g is supposed to be regular except at the points of a sequence $\{iy_n\}$, $0=y_0<y_1<\cdots$ where poles and logarithmic terms may occur. The author wishes to compare the Laplace transforms

$$G(p)=\text{finite part} \int_0^{\infty} g(t)e^{-pt}dt,$$

$$G_1(p)=\lim_{b\to\infty}\text{finite part} \int_0^b g(it)e^{-pt}dt.$$

He states conditions under which

$$G_1(p)=-iG(-ip)-\pi(\tfrac{1}{2}r_0+\sum_{n\geq1}r_n)$$

for $\operatorname{Re} p>0$, $\operatorname{Im} p>0$. Here r_n is the residue of $g(z)e^{ipz}$ at $z=iy_n$. Several examples are given. *J. Korevaar.*

Mehra, A. N. On Meijer transform of two variables. Bull. Calcutta Math. Soc. 48 (1956), 83–94.

Let us set

$$K_1(p,x)=pe^{-\frac{1}{2}px}W_{k_1+\frac{1}{2},\,m_1}(px)(px)^{-k_1-\frac{1}{2}},$$

$$K_2(q,y)=qe^{-\frac{1}{2}qy}W_{k_2+\frac{1}{2},\,m_2}(qy)(qy)^{-k_2-\frac{1}{2}}.$$

The author introduces the following integral transformation

$$\phi(p,q)=\int_0^{\infty}\int_0^{\infty}K_1(p,x)K_2(q,y)f(x,y)dxdy,$$

and studies its elementary properties. A number of particular cases are computed. *I. I. Hirschman, Jr.*

Blackman, Jerome. Convolutions with rational kernels. Proc. Amer. Math. Soc. 8 (1957), 100–106.

An explicit inversion formula is given for the convolution transform

$$(*)\qquad f(u)=\int k(u-x)d\alpha(x)$$

in the case when the kernel $k(x)$ is a rational function having no singularities on the real axis and vanishing at infinity. It is further assumed that the Fourier transform of $k(x)$ has no real zeroes.

It is assumed that $\alpha(x)$ is locally of bounded variation, and that the integral (*) exists for at least one value of u near the real axis. Unfortunately, neither necessary nor sufficient conditions for the given function $f(u)$ to come from such an α are stated.

The equation (*) has an obvious formal solution in terms of Fourier transforms. The chief point of the paper is to define a type of summability (akin to Gauss summability) for which this formal solution can be justified.
H. F. Weinberger (Madison, Wis.).

See also: Browder, p. 902; Hellwig, p. 903; Sofronov, p. 906; Choudhury, p. 963; Hill, p. 966; Scott, p. 968; Aĭzerman, p. 981.

Ordinary Differential Equations

Heilbronn, Georges. Intégration des équations différentielles ordinaires par la méthode de Drach. Mémor. Sci. Math., no. 133. Gauthier-Villars, Paris, 1956. 103 pp. 1300 francs.

C'est une lucide exposition de l'intégration logique de J. Drach [cf. en particulier, Proc. 5th Internat. Congress Math., Cambridge, Eng., 1912, v. 1, Cambridge, 1913, pp. 438–497; Ann. Sci. Ecole Norm. Sup. (3) 37 (1920), 1–94].

Table des matières: I. L'intégration logique; II. Equations du premier ordre. III. Equations du second ordre. IV. Equations linéaires. Bibliographie.

L'auteur a également donné quelques contributions à la théorie de Drach. La bibliographie est incomplète.
D. S. Mitrinovitch (Belgrade).

Azbelev, N. V.; and Tonkov, L. V. Theorem about the estimation of error of an approximate solution of a differential equation. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 515–516. (Russian)

The problem considered is to estimate the deviation of a given function $u(x)$ from the exact solution $y(x)$ of an equation

$$y^{(n)}=f(x, y, y', \cdots, y^{(n-1)}),$$

satisfying prescribed initial conditions. The authors give a procedure based on the use of comparison functions which are solutions of auxiliary linear equations. The procedure leads to an estimate of all derivatives of the difference $\eta=u-y$ up to the order $(n-1)$ in an interval whose length can be estimated from the behavior of a solution of a homogeneous auxiliary equation.
R. Finn (Pasadena, Calif.).

Bielecki, A. Remarques sur la méthode de T. Ważewski dans l'étude qualitative des équations différentielles ordinaires. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 493–495.

The present paper and the two reviewed below concern Ważewski's topological theory of the asymptotic behavior of solutions of differential systems, and are particularly connected with the papers of T. Ważewski [Ann. Soc. Polon. Math. **20** (1947), 279–313; Bull. Acad. Polon. Sci. Cl. III. **3** (1955), 143–148; MR **10**, 122; **16**, 1109], F. Albrecht [ibid. **2** (1954), 315–318; MR **16**, 248], A. Pliś [ibid. **2** (1954), 415–418; MR **16**, 700]. As usual let $\omega C \Omega$ denote open sets in the (t, x)-space E_{n+1}, $x=(x_1, \cdots, x_n)$, and suppose that for every point $Q \in \omega$ there passes one and only one solution of the system $dx/dt=f(t, x)$, where f is a continuous vector function in Ω. The points of egress and of strict egress of the frontier ϕ of ω in Ω have been already defined in the above quoted papers. By relaxing somewhat the requirements for a strict egress, the author defines the points of strong egress: namely a point $R \in \phi$ is said to be of strong egress if R is a point of egress and there are no integrals $x=\varphi(t)$, $\alpha \leqq t \leqq \beta$, $\alpha<\beta$, with $(\alpha, \varphi(\alpha))=R$, $(t, \varphi(t)) \in \bar{\omega}$ for $\alpha \leqq t \leqq \beta$. All points of strict egress are points of strong egress and examples show that there are points of strong egress which are not points of strict egress. The following lemma extends a previous one of T. Ważewski [see the second paper cited above]: If all points of egress are points of strong egress then the consequent $R=C(Q)$ of the point Q is a continuous function in $W+S$, where S is the set of points of egress and W is the left shadow of S (and thus $C(Q)$ defines a retraction of $W+S$ into S). *L. Cesari.*

Bielecki, A. Sur une méthode de régularisation des équations différentielles ordinaires dont les intégrales ne remplissent pas la condition d'unicité. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 497–501.

In application of the paper reviewed above the following theorem, among others, is proved, which is analogous to a previous one of T. Ważewski [Ann. Soc. Polon. Math. **20** (1947), 279–313; MR **10**, 122] and where no uniqueness theorem is required. Let $\omega C \Omega$ denote open sets of the (t, x)-space E_{n+1}, f a continuous vector function in Ω, ϕ the frontier of ω in Ω, $SC\phi$ the set of the points of egress for the differential system $dx/dt=f(t, x)$, A the set covered by the semi-integrals $L^+C\omega$. Suppose that $\phi-S$ is an F_σ-set, and the set $S''=S-S'$ is closed, where $S'CS$ is the set of the points of strong egress. If V is a subset of $\omega+S'$ and VS' is a retraction of S', and VS' is not a retraction of V, then there exists a semi-integral L^+ in $\omega+S''$ which either encounters S'', or is an asymptotic integral in ω. *L. Cesari* (Lafayette, Ind.).

Bielecki, A. Certaines propriétés topologiques des intégrales des équations différentielles ordinaires. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 503–506.

In the same line of the two papers reviewed above, the author proves theorems analogous to previous ones of T. Ważewski [same Bull. **3** (1955), 143–148; MR **16**, 1109]. These new statements concern cases where a uniqueness theorem does not necessarily hold. Assume $UC\omega+S$ is a set closed relatively to $\omega+S$ containing all semi-integrals L^- in $\omega+S$ issued from points of U and all semi-integrals L^+ in Ω issued from points of U, and put $B=\omega-U$, $S'=S-U$. Among other statements conditions are given, too involved to be repeated here, in order that $B+S'$ and $S\times I$, $I=[0 \leqq u \leqq 1]$, are homeomorphic. *L. Cesari.*

Albrecht, F. Un théorème de comportement asymptotique des solutions des systèmes d'équations différentielles. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 737–739.

As a further contribution to Ważewski's topological method [same Bull. **3** (1955), 143–148; MR **16**, 1109] the author proves the following theorem which extends a previous one of the same author and whose novelty consists in the fact that points of "internal contact" are allowed. As usual let GCD be open sets of the (t, x)-space R_{n+1}, and $dx/dt=f(t, x)$ be a differential system where f is a continuous vector function in D and for which a theorem of uniqueness holds. A point p of the frontier ϕ of G in D is said to be of internal contact if a solution $x=\varphi(t)$, $t_0-\varepsilon \leqq t \leqq t_0+\varepsilon$, $\varepsilon>0$, exists such that $(t_0, \varphi(t_0))=p$, $(t, \varphi(t)) \in G$ for all $t \neq t_0$, $t_0-\varepsilon \leqq t \leqq t_0+\varepsilon$. Suppose that the set of the points of egress is the sum $S+A$ of the set S of all points of strict egress and of the set A of the points of internal contact, and A is closed. Suppose that a components S' of S has the property $AL^-(p)=0$ for all $p \in S'$, where L^- is the semi-integral issued from p. Suppose that there exists a connected set Z such that $ZCGS$, $ZG \neq 0$, $ZS' \neq 0$, and ZS is a retraction of S and ZS is not a retraction of Z. Then there exists a point $p_0 \in ZG$ such that $L^+(p_0)C\bar{G}$. *L. Cesari.*

Minc, R. M. On the character of equilibrium of a system of three differential equations in the case when one of the roots of the characteristic equation equals zero. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 535–537. (Russian)

An analytical system in three variables with an isolated singularity at the origin may be reduced to the form

$$\begin{aligned}\dot{x}&=a_1x+b_1y+X(x, y, z)\\ \dot{y}&=a_2x+b_2y+Y(x, y, z)\\ \dot{z}&=\qquad\qquad\quad Z(x, y, z)\end{aligned}$$

where X, Y, Z are power series beginning with terms of degree at least two. It is assumed here that the characteristic roots λ_1, λ_2 of $\left\|\begin{matrix}a_1 b_1\\ a_2 b_2\end{matrix}\right\|$ are $\neq 0$. With Lyapunov one observes that the system $\dot{x}=\dot{y}=0$ has then a solution $x(z)$, $y(z)$ holomorphic and zero at $z=0$. Set

$$f(z)=R_3(x(z), y(z), z)=\Delta z^m+\cdots.$$

The author examines the various sign possibilities for λ_1, λ_2, together with the size and sign of Δ and the parity of m and describes in three theorems the local phase-portraits at the origin. There are only indications of proofs.

S. Lefschetz (Mexico City).

Reid, William T. Oscillation criteria for linear differential systems with complex coefficients. Pacific J. Math. 6 (1956), 733–751.

This paper deals with a self-adjoint second order linear vector differential equation

$$L[u]=(R(x)u'+Q(x)u)'-(Q^*(x)u'+P(x)u)=0$$

with complex-valued coefficients. Various criteria, general as well as specific, for oscillation and nonoscillation are given. These criteria either correspond to or are direct generalizations of known results for a real system or for a single equation of the second order. These results are also applied to obtain sufficient conditions for nonoscillation for a general vector differential equation

$$A_2(x)u''+A_1(x)u'+A_0(x)u=0.$$

Choy-tak Taam (Washington, D.C.).

Nikolenko, L. D. On a sufficient condition for non-oscillating character of solutions of the equation $y''+f(x)y=0$. Dokl. Akad. Nauk SSSR (N.S.) 110 (1956), 929–931. (Russian)

Es wird der folgende Satz bewiesen: Wenn

$$\int^\infty L_{p+1}(x)\varphi_p(x)dx<\infty,$$

wo $L_p(x)=x\ln x\ln\ln x\cdots\ln^{(p)}x$, $\ln^{(p)}x=\ln\ \ln^{(p-1)}x$, $\mathrm{n}^{(0)}\,x=x$, und

$$\varphi_p(x)=f(x)-\tfrac{1}{4}\sum_{i=0}^{p}[L_i(x)]^{-2}\text{ bei }f(x)-\tfrac{1}{4}\sum_{i=0}^{p}[L_i(x)]^{-2}\geqq0,$$

$$\varphi_p(x)=0\qquad\text{bei }f(x)-\tfrac{1}{4}\sum_{i=0}^{p}[L_i(x)]^{-2}<0,$$

dann ist jede nichttriviale Lösung von (1) $y''+f(x)y=0$ nichtoszillatorisch. Zu diesem Satz siehe die Bemerkung von Atkinson in MR 17, 263 zu einer früheren Arbeit des Autors, wo der Satz für $p=0$ bewiesen wurde.

M. Zlámal (Brno).

Colautti, Maria Pia. Sulla maggiorazione "a priori" delle soluzioni delle equazioni e dei sistemi di equazioni differenziali lineari ordinarie del secondo ordine. Matematiche, Catania 11 (1956), 8–99.

La memoria è dedicata all'ottenimento di formole di maggiorazione a "priori" per le equazioni e i sistemi di equazioni differenziali ordinarie lineari del secondo ordine.

I risultati di maggiore interesse sono del tipo seguente. Sia **E** una matrice operazionale di ordine m, i cui elementi sono operatori differenziali ordinari lineari del secondo ordine a coefficienti variabili, **u** un vettore a m componenti $u_1, u_2, \cdots, u_m$ continue e dotate di derivate prime e seconde continue in un intervallo (x_1, x_2) verificanti agli estremi di (x_1, x_2) condizioni del tipo seguente:

$$\alpha_{ij}u_i(x_j)+\beta_{ij}u_i'(x_j),$$

con α_{ij} e β_{ij} opportune costanti ($i=1, 2, \cdots, m$; $j=1, 2$). Vengono dimostrate formole di maggiorazione di questo tipo:

$$\int_{x_1}^{x_2}|\mathbf{u}|^2dx\leqq K\int_{x_1}^{x_2}|\mathbf{E}*\mathbf{u}|^2dx$$

e viene data una valutazione numerica effettiva della costante K.

Il caso $m=1$ è considerato in maggior dettaglio e vengono ottenute formole di maggiorazione con normalizzazioni di tipo $\mathcal{L}^{(p)}$ ($p\geqq2$) dalle quali, con passaggio al limite per $p\to\infty$, si ottengono, sfruttando un noto lemma di Riesz, maggiorazioni puntuali.

Come applicazione, nel caso $p=2$ ed $m\geqq1$, vengono ottenute limitazioni inferiori per lo spettro dei problemi ai limiti di Sturm-Liouville. *G. Fichera* (Roma).

Perčinkova-V'čkova, D. Sur une équation différentielle linéaire du second ordre. Bull. Soc. Math. Phys. Macédoine 6 (1955), 27–29. (Macedonian. French summary)

On démontre que l'équation différentielle linéaire

$$x^2(1+x^3)y''-\tfrac{3}{2}xy'+y=0,\ y=y(x),$$

a comme solutions particulières les fonctions suivantes:

$$y_1=-3t^2/(t^3+2),\ y_2=-3(t^4+8t)^{\frac{1}{2}}/(t^3+2),$$

avec $x=-3t/(t^3+2)$. *D. S. Mitrinovitch* (Belgrade).

Loud, W. S. Some growth theorems for linear ordinary differential equations. Trans. Amer. Math. Soc. 85 (1957), 257–264.

The author considers the problem of obtaining best possible exponential bounds for the growth of solutions of linear differential systems of the form $x'=A(t)x$. After obtaining a result for general systems, he turns to a study of the second order equation $u''+a(t)u=0$. As a consequence of these results, he obtains bounds for the Floquet exponent associated with periodic $a(t)$. *R. Bellman.*

Kaščeev, N. A. Precise limit of applicability of S. A. Čaplygin's theorem for a linear equation. Dokl. Akad. Nauk SSSR (N.S.) 111 (1956), 937–940. (Russian)

Consider the equation

$$(1)\qquad L[y]=y^{(n)}-\sum a_i(x)y^{(n-i)}=f(x),$$

where the a_i and f are continuous on a segment $[a-b]$. Let $u(x)$ be a comparison function n-fold differentiable and satisfying

$$(2)\qquad u^{(i)}(x_0)=y_0^{(i)}\ \ (i\leqq n-1),$$

where $y(x)$ is the solution of (1) with initial values $y_0^{(i)}$ at $x_0\geqq a$. Suppose that

$$(3)\qquad L[u(x)]-f(x)\geqq0\ (x_0<x\leqq X).$$

The value x_k^* is said to be the exact applicability limit of Čaplygin's theorem of order k for (1) at x_0, wherever for any $u(x)$ as above with $x_0<x\leqq b$, we have $u^{(k)}(x)=y^k(x)$ in $x_0<x<x_k^*$, where this ceases to hold if one replaces x_k by any larger number. Let $\varphi(x, \alpha)$ be the solution of $L[y]=0$ such that $\varphi_x^{(i)}(\alpha, \alpha)=0$, $i<n-1$, $\varphi_x^{(n-1)}(\alpha, \alpha)=1$. Theorem. $x_k^*=\sup X$, $X>x_0$ for which $\varphi_x^{(k)}(x, \alpha)\geqq0$ in the triangle bounded by $\alpha=x$, $\alpha=x_0$, $x=X$.

It is shown that for any $L[y]$ there exists a $\lambda>0$ such that for any $x_0\leqq x\leqq b$ and any $u(x)$ satisfying (3) (with $>$ instead of $\geqq$) we have $u(x)>y(x)$ on $x_0<x\leqq x_0+\lambda$.

S. Lefschetz (Mexico City).

des Cloizeaux, Jacques. Spectre de fréquences d'une chaîne linéaire désordonnée. J. Phys. Radium (8) **18** (1957), 131–132.

Given a system of linear differential equations of the form $du_j/dt=r_jy_{j+1}-r_{j-1}u_{j-1}$, where the r_i are positive random variables with a common distribution, the problem is to deduce the asymptotic behavior of the frequency spectrum as the degree of the system increases arbitrarily. This problem was resolved by Dyson [Phys. Rev. (2) **92** (1953), 1331–1338; MR **15**, 492] using the method of moments, and by the reviewer [ibid. **101** (1956), 19; MR **17**, 930] using continued fractions and recurrence relations. The author introduces a method based upon phases which bears a resemblance to the continued fraction method mentioned above. *R. Bellman.*

Gheorghiu, N. Solutions presque-périodiques et asymptotiquement presque-périodiques de quelques équations différentielles non linéaires du premier ordre. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **1** (1955), 17–20. (Romanian. Russian and French summaries)

Relativement à l'équation différentielle

(1) $$dx/dt=f(x, t)$$

on démontre deux propositions suivantes:

I. Si: 1° $f(x, t)$ est une fonction définie et continue pour $-\infty<t<+\infty$ et $|x|\leqq M$; 2° $f_x(x, t)$ est continue dans la même bande et $f_x(x, t)\geqq k>0$ ou $f_x(x, t)\leqq k<0$; 3° $f(x, t)$ est une fonction presque-périodique de t, uniformément par rapport à x, dans la bande considéré — alors toute solution bornée de (1) est presque-périodique.

II. Si la fonction $f(x, t)$ remplit les conditions indiquées 1° et 2°, dans la demi-bande $t\geqq\alpha$, $|x|\leqq M$, étant asymptotiquement presque périodique de t (au sens de Fréchet), uniformément par rapport à x, alors toute solution bornée de (1) est asymptotiquement presque-périodique.

La démonstration est fondée sur la linéarisation de (1), au moyen d'un lemme dû à Hadamard.

Le rapport est écrit d'après le résumé.

D. S. Mitrinovitch (Belgrade).

Reissig, Rolf. Über die Existenz periodischer Lösungen für Differentialgleichungen 2. Ordnung mit einem Störungsglied. Math. Nachr. **14** (1955), 341–348 (1956).

The equation $\ddot{x}+f(x, \dot{x})\dot{x}+g(x)=e(t)$, $e(t+L)=e(t)$, is treated. The results of the reviewer [J. Math. Phys. **22** (1943), 41–48; MR **5**, 66] are considerably improved. Let $\dot{x}=v$. Then for $|x|\leqq a$ or $|v|\leqq b$ the author assumes $f(x, v)\geqq-f$, where f is a positive constant. Otherwise $f(x, v)\geqq0$. If $|e(t)|\leqq M$ and $xg(x)\geqq0$ then let $f(x, y)|v|\geqq|g(x)|+M$ for $x\leqq-a$, $v\geqq c>b$ and for $x\geqq a$, $v\leqq-c$. For $|x|\geqq a$, $|g(x)|\geqq M+bf+\delta$, where $\delta>0$. These conditions suffice for the existence of a periodic solution of period L. More general conditions are given. *N. Levinson.*

Gorbunov, A. D. Estimates of characteristic exponents of solutions of a system of ordinary linear homogeneous differential equations. Vestnik Moskov. Univ. **11** (1956), no. 2, 7–13. (Russian)

Given a homogeneous linear system (*) $x'=A(t)x$, $x=(x_1, \cdots, x_n)$, whose coefficients are real continuous bounded functions of t in $[0, +\infty)$, the author considers, in line with his previous paper [same Vestnik 1950, no. 10, 19–26; MR **14**, 751], a quadratic form

$$G(t, x)=\sum A_{ih}(t)x_ix_h\ (A_{ih}=A_{hi}),$$

which is positive definite for $t\geqq0$, and whose coefficients are continuous bounded real functions of t in $[0, +\infty)$. If $g(t, x)$ denotes the usual derivative of G written in terms of the system (*), and $n_G(\tau)$, $N_G(\tau)$ denote the minimum and the maximum of $g(\tau, x)$ for all real x with $G(\tau, x)=1$, then the author proves the following estimates of the Lyapunov type numbers $\pi[x(t)]$ of the nontrivial solutions $x(t)$ of (*):

(a) $$2\pi[x(t)]\geqq \limsup t^{-1}\int_0^t n(\tau)d\tau,$$

(b) $$2\pi[x(t)]\leqq\pi[\omega]+\limsup t^{-1}\int_0^t N_G(\tau)d\tau,$$

where lim sup are taken as $t\to+\infty$, and $\omega(t)$ is the vector of the n principal minors of $A(t)=[A_{ih}]$ divided by the determinant of $A(t)$. Use is made of results of the quoted paper by the same author. *L. Cesari* (Lafayette, Ind.).

Mel'nikov, G. I. Certain questions of the direct method of Lyapunov. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 326–329. (Russian)

The present paper is in the line of Lyapunov's second method. If a definite positive function V has been determined which, with the sign of the usual derivative dV/dt assures the asymptotic stability of the zero solution of a given linear or nonlinear differential system, the question has been raised to evaluate the damping of the motion around the origin in the phase space [see A. D. Gorbunov, Moskov. Gos. Univ. Uč. Zap. **165**, Mat. **7** (1954), 39–78, MR **16**, 475, for linear systems, and N. G. Četaev, Prikl. Mat. Meh. **15** (1951), 371–372, MR **13**, 346, for nonlinear systems]. Some new results on the same effect are briefly stated without proofs in the present paper. For instance, if a definite positive function $V(x_1, \cdots, x_n, t)$ has been found having an infinitesimal upper bound for $t\geqq t_0$ and $dV/dt\leqq f(t)V+\varphi(t)V^{1+k}$ for $0\leqq V\leqq h$, where h, k are constant and $f(t)$, $\varphi(t)$ given functions then an evaluation is given of the time required for the point to go from the set $V=h_1$ to the set $V=\varepsilon$, for given $0<\varepsilon<h_1<h$, ε, h_1 small. *L. Cesari.*

Pliss, V. A. Investigation of a non-linear differential equation of third order. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 1178–1180. (Russian)

It is known [Erugin, Prikl. Mat. Meh. **14** (1950), 459–512; **16** (1952), 620–628; MR **12**, 412; **14**, 376] that systems of two equations of Aizerman type [Uspehi Mat. Nauk (N.S.) **4** (1949), no. 4(32), 187–188; MR **11**, 177] have no periodic solutions. A number of studies have been made of such systems of three equations, but they always lacked periodic solutions. Such a system is now brought out possessing periodic solutions under certain conditions. It arises out of

(1) $$\dddot{\xi}+f(\ddot{\xi})+\dot{\xi}+\xi=0,$$

where $f(\eta)$ is continuous and satisfies a Lipschitz condition for all η,

(2) $$f(0)=0,\ \eta f(\eta)>\eta^2 \text{ for } \eta\neq0.$$

Instead of (1) the author studies the equivalent system

(3) $$\dot{x}=y-f(x),\ \dot{y}=z-x,\ \dot{z}=-x$$

and states (without proofs) many properties of the trajectories of (3) especially in relation to crossing $x=0$. Noteworthy is this: Let P be the domain $x=0$, $y>0$, $z>0$, $y^2+z^2<R$ (a certain R). If there exists $x_0\geqq0$ and

$\varepsilon>0$ such that $f'(x)>1+\varepsilon$ for $x\geq x_0$, then a necessary and sufficient condition for the stability of the origin in the large is the non-existence of a periodic solution cutting P in just one general point. An example is also given of an $f(x)$ for which there is a periodic solution.

S. Lefschetz (Mexico City).

Popov, E. P. A generalization of the asymptotic method of N. N. Bogolyubov in the theory of non-linear oscillations. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 308–311. (Russian)

The differential equation

$$(1)\qquad \ddot{x}+2b\dot{x}+c^2x=\varepsilon f(x,\dot{x}),$$

where b, c, ε are positive constants (ε small) and f is non-linear, is subjected to the Bogoliubov general method [Bogoliubov and Mitropolski, Asymptotic methods in the theory of non-linear oscillations, Gostehizdat, Moscow, 1955, Ch. 1; MR **17**, 368]. This method was applied in that book to the case $b=0$. Let $\omega=\sqrt{(c^2-b^2)}$. A solution is assumed of the form

$$x=a\cos\psi+\varepsilon\varphi_1(a,\psi)+\cdots,$$

with

$$\dot{a}=-ba+\varepsilon\Phi_1(a)+\cdots,$$
$$\dot{\psi}=\omega+\varepsilon B_1(a)+\cdots,$$

the series being taken merely "formally" and used à la Bogoliubov only to find approximate solutions mod ε^m.

S. Lefschetz (Mexico City).

Ryabov, Yu. A. Complement to the article "Generalization of a theorem of A. M. Lyapunov". Moskov. Gos. Univ. Uč. Zap. **181**, Mat. 8 (1956), 241. (Russian)

The author has given [same Zap. **165**, Mat. 7 (1954), 131–150; MR **16**, 474] a generalization of a certain theorem of A. M. Lyapunov. Erugin has pointed out that the proof given there holds only when all the numbers σ loc. cit. are $\neq 0$. The author corrects his proof showing that it holds even when some of the σ's are zero.

S. Lefschetz (Mexico City).

Demidovič, B. P. On the existence of a limiting regime of a certain non-linear system of ordinary differential equations. Moskov. Gos. Univ. Uč. Zap. **181**. Mat. 8 (1956), 3–12. (Russian)

A finite symmetric matrix $A=(a_{ij})$ is positive [negative] definite whenever its characteristic roots λ_i [the $-\lambda_i$] are all positive. It is uniformly positive [negative] definite (=u.p.d., u.n.d.) whenever the λ_i [the $-\lambda_i$] are above a certain $h>0$.

Consider a system

$$(1)\qquad \dot{X}=F(X)+G(t),$$

where X, F, G are real n-vectors, F is of class C^1 in the components $x_1, \cdots, x_n$ of X for all values of these variables and $G(t)$ is continuous with period T. The author discussed above all the existence of a periodic solution to which tend all the other solutions.

Let $f_i(X)$ be the components of F and Ξ_i an n-vector with components ξ_{ij}. Consider the matrix $W=(f_{ixj}(\Xi_i))$ and the related symmetric matrix $J(F)=\frac{1}{2}(W+W')$. Theorem 1. If $J(F)$ is u.p.d. or u.p.n. then (1) has a solution $X_0(t)$ of period T, and for $t\to\pm\infty$ all solutions tend to $X_0(t)$. Hence $X_0(t)$ is unique.

Suppose now that in (1) $G(t)$ is merely bounded for all t. Theorem 2. Under the same assumptions for F there exists then a bounded solution $X_1(t)$ to which all other solutions tend asymptotically and $X_1(t)$ is again unique.

S. Lefschetz (Mexico City).

★ **Cartwright, M. L. Some aspects of the theory of nonlinear vibrations.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III. pp. 71–76. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. \$7.00.

This is an expository article in which the author reviews some known results as they pertain to two general problems; viz., that of determining conditions which preclude undesirable oscillations in a nonlinear system and that of determining how rapidly a given nonlinear system approaches its steady state.

C. E. Langenhop (Ames, Iowa).

Massera, José L. On the stability of spaces of infinite dimension. Rev. Un. Mat. Argentina **17** (1955), 135–147 (1956). (Spanish)

The author considers a differential equation $x'=f(x,t)$ in which x and f have values in a Banach space. It is proved that, if $x\equiv 0$ is a uniformly asymptotically stable solution, then there exists a (generalized) Lyapunov function $V(x,t)$ for the differential equation. As a preparation, the author also proves theorems on dependence of solutions on initial conditions and on parameters.

W. Kaplan (Ann Arbor, Mich.).

Barbălat, I.; et Halanay, A. Un critère d'existence d'un cycle limite stable pour l'équation des oscillations non linéaires. Acad. R. P. Romîne. Stud. Cerc. Mat. **7** (1956), 81–94. (Romanian. Russian and French summaries)

The equation $x''+f(x)x'+g(x)=0$ is shown to have a stable limit cycle in the (x, x') plane, an analogue in certain cases not covered by Filippov. Here $xg(x)>0$ for $x\neq 0$ and g and f are continuous. The auxiliary functions $z=\operatorname{sgn} x\int_0^x g(\xi)d\xi$ and $\phi(z)=\int_0^{x(z)} f(\zeta)d\xi$ are introduced and the criteria developed in terms of these functions. A uniqueness criterion is also given which is called an analogue of that of Levinson and Smith.

N. Levinson (Cambridge, Mass.).

★ **Pinney, Edmund. Nonlinear differential equations systems.** Contributions to the theory of nonlinear oscillations, vol. 3, pp. 31–56. Annals of Mathematics Studies, no. 36. Princeton University Press, Princeton, N. J., 1956. \$4.00.

The purpose of this paper, as stated in its introduction, is to give a practical method for making detailed investigations of particular systems of nonlinear differential equations. The method provides approximate solutions to the initial value problem: $\dot{x}=Ax+\varepsilon f(x,t)$, $x(0)=x_0$, for small ε, over a fixed range $0\leq t\leq T$. The details are too complicated to be described here: the principle of the method, as far as the reviewer can see it, is as follows. First make a suitable change of variable, such as $x=e^{At}y$, to bring the system to the form $\dot{y}=\varepsilon g(v,t)$. Next, by picking out the dominant terms in this equation, write down an approximate solution y_1 and calculate an explicit upper bound, say $|y_1|\leq K'(\varepsilon)$. If an upper bound $|y|\leq K$ is assumed for the exact solution, and if $y=y_1+y_2$, one can find an upper bound $|y_2|\leq E(\varepsilon,K)$ for the error y_2 and hence deduce the bound $|y|\leq K'(\varepsilon)+E(\varepsilon,K)$. If this bound does not exceed K, the original assumption $|y|\leq K$

has been confirmed. Thus the method will succeed if K can be chosen so that $K'(\varepsilon)+E(\varepsilon, K)\leq K$ and so that $E(\varepsilon, K)$ is of smaller order of magnitude than the approximate solution y_1; the method will fail if, for any reasonable choice of K, either $K'(\varepsilon)+E(\varepsilon, K)>K$ or $E(\varepsilon, K)$ is of larger order than y_1. *G. E. H. Reuter.*

Hartman, Philip; and Putnam, Calvin R. The essential spectra belonging to bounded and half-bounded potentials. Duke Math. J. 23 (1956), 561–570.

It is assumed $f(t)$ is continuous on $[0, \infty]$ and that $x''+(\lambda+f(t))x=0$ is in the limit point case. The essential spectrum S' is considered. The complement of the closed set S' is a sum of open intervals $\sum (\lambda_k, \lambda^k)$. Estimates in the size of $\lambda^k-\lambda_k$ are obtained. The results are an improvement in earlier results of the authors [Amer. J. Math. 72 (1950), 849–862; J. London Math. Soc. 27 (1952), 492–496; MR 12, 414; 14, 278]. Previous estimates of $\lambda^k-\lambda_k$ involved a main term and a correction term and it is shown now that the latter can be omitted. One result is that if for some $\delta>0$

$$\liminf_{T\to\infty} T^{-1}\int_0^T \max_{|h|\leq\delta} |f(t+h)-f(t)|dt=0,$$

then S' contains a half-line $C\leq\lambda<\infty$. The procedure involves getting improved estimates on $N(T, \lambda)$, the number of zeros of a real solution, $x(t, \lambda)$ on $0<t\leq T$.

N. Levinson (Cambridge, Mass.).

Kamynin, L. I. On Cauchy's problem for an infinite system of ordinary differential equations. Dokl. Akad. Nauk SSSR (N.S.) 109 (1956), 446–449. (Russian)

This note is concerned with the existence and the uniqueness of the solution of Cauchy's problem for an infinite system of ordinary differential equations of the form $\partial u(x, t)/\partial t=f(x, t; \cdots, u(x+kh, t), \cdots)$, $k=\cdots, -2, -1, 0, 1, 2, \cdots$; the initial condition being $u(x, 0)=\alpha(x)$, $-\infty<x<+\infty$. The author [same Dokl. (N.S.) 82 (1952), 13–16; 93 (1953), 397–400; 95 (1954), 13–16; 103 (1955), 545–548; MR 14, 172; 16, 251, 524; 17, 268] has previously considered these questions in the case when the function f depends only on a finite number of the unknown functions $u(x, t)$. Here, the function f may depend on infinitely many of the $u(x, t)$. Sufficient conditions for the existence and uniqueness of the solution are given. These conditions involve the growth of the functions f and u. The sufficient conditions for the existence of a solution of the Cauchy problem for the system $du_n(t)/dt = \sum_{k=-\infty}^{+\infty} c_k u_{n+k}(t)$ ($n=\cdots, -2, -1, 0, 1, 2, \cdots$) [given by V. I. Protasov, ibid. 105 (1955), 218–221; MR 17, 847] follow as a special case. *J. B. Diaz* (College Park, Md.).

See also: Lewy, p. 883; Pliss, p. 899; Čermák, p. 905; Coddington, p. 915; Maslov, p. 916; Tihonov and Samarskiĭ, p. 938; Merkin, p. 960; Sharma, p. 963; Keller, p. 964; Chu and Herrmann, p. 964; Berezkin, p. 981.

Partial Differential Equations

Ladyženskaya, O. A. On the construction of discontinuous solutions of quasi-linear hyperbolic equations as limits of solutions of the corresponding parabolic equations when the "coefficient of viscosity" tends toward zero. Dokl. Akad. Nauk SSSR (N.S.) 111 (1956), 291–294. (Russian)

Es handelt sich um die Konstruktion von Lösungen des Cauchyschen Problems für die quasilineare Differentialgleichung

$$(*)\qquad \frac{\partial\varphi(x, t, u)}{\partial t}+\frac{\partial\varphi(x, t, u)}{\partial x}=0,\quad u|_{t=0}=u_0(x),$$

in einem beliebig großen Definitionsbereich (x, t). Dabei müssen von vornherein unstetige Lösungen zugelassen werden. In der Hydrodynamik ergänzt man in solchen Fällen (*) zweckmäßig durch ein die Viskosität zum Ausdruck bringendes Glied und versucht mit den Lösungen (u^ε) der so modifizierten Gleichung einen Grenzübergang zum viskositätsfreien Fall (u^0). —Die Verfasserin beweist zunächst die Konvergenz dieses Grenzüberganges $(\varepsilon\to 0, u^\varepsilon\to u^0)$. Sodann werden Eigenschaften der Grenzlösung gefunden, durch welche diese eindeutig bestimmt ist. In der Beweisführung wird φ zunächst von t und x unabhängig angenommen. $\varphi(u)$ wird dann zweimal stetig differenzierbar vorausgesetzt und im Intervall $[-c_1, c_1]$ der Bedingung

$$\varphi_{uu}(u)>\alpha>0$$

unterworfen. Sowohl u_0 wie auch $\varphi, \varphi_u, \varphi_{uu}$ können durch hinreichend glatte Funktionen $u_0{}^n$ approximiert werden. — Sodann wird die Gleichung

$$\frac{\partial u}{\partial t}+\frac{\partial\varphi_u(u)}{\partial x}=\varepsilon\frac{\partial^2 u}{\partial x^2},\quad u|_{t=0}=u_0{}^n(x)$$

behandelt. Für die bekannte beschränkte Lösung u^ε gilt

$$\max_{0\leq t\leq T}\frac{\partial u^\varepsilon{}_{n,m}}{\partial x}<c_3(T)^{n,m},$$

wobei $c_3(T)$ nicht von ε abhängt. Ferner gilt der Satz: die $u^\varepsilon{}_{n,m}$ konvergieren mit $\varepsilon\to 0$ und $m,n\to\infty$ gegen die Grenzfunktion $u(x, t)$. Diese Grenzlösung genügt der Identität

$$\int_{-\infty}^{+\infty}\int_0^\infty [u\Phi_t+\varphi(u)\Phi_x]dxdt+\int_{-\infty}^{+\infty} u_0\Phi(x, 0)dx=0,$$

in welcher $\Phi(x, t)$ auf $t\geq 0$ beschränkt und beliebig stetig differenzierbar vorgegeben werden kann. Nach einem Ergebnis von O. W. Gussewa existiert innerhalb $u^\varepsilon{}_{n,m}$ eine Teilfolge, die fast überall gegen u konvergiert. Zum Schluß wird noch der erwähnte Eindeutigkeitssatz bewiesen. *M. Pinl* (Köln).

Vvedenskaya, N. D. Solution of the Cauchy problem for a non-linear equation with discontinuous initial conditions by the method of finite differences. Dokl. Akad. Nauk SSSR (N.S.) 111 (1956), 517–520. (Russian)

As is known, a solution of the equation

$$(1)\qquad \frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\phi(t, x, u)=0$$

subject to the initial condition $u(0, x)=u_0(x)$ ($u_0(x)$ a smooth function) can always be found in a sufficiently small neighborhood of the line $t=0$. Oleĭnik [same Dokl. (N.S.) 95 (1954), 451–454; MR 16, 253] proved that under suitable assumptions on $\phi(t, x, u)$, notably that $\phi_{uu}''>c>0$ for $t\geq\tau>0$, a generalized solution $u(t, x)$ of (1) exists in any region $G\{-\infty<x<\infty, T\geq t\geq 0\}$. Oleĭnik also proved [Uspehi Mat. Nauk (N.S.) 9 (1954), no. 3(61), 231–233] that a solution of

$$(2)\qquad \varepsilon\frac{\partial^2 u_\varepsilon}{\partial x^2}=\frac{\partial u_\varepsilon}{\partial t}+\frac{\partial}{\partial x}\phi(t, x, u_\varepsilon),\ u_\varepsilon(0, x)=u_0(x),$$

tends to $u(t, x)$ in an average sense as $\varepsilon\to 0$. These results

are extended by the author, who obtains $u(t, x)$ as the limit of solutions of a finite difference scheme proposed by Lax [Comm. Pure Appl. Math. **7** (1954), 159–193; MR **16**, 524], as the mesh size tends to zero appropriately. The author obtains also the new result, that by proper choice of the mesh ratio the solution of the finite difference scheme can be made to converge to a generalized solution of (2). As a final result, the author proves the convergence to $u(t, x)$ of the solutions of another difference scheme, due to Godunov. *R. Finn* (Pasadena, Calif.).

Ballabh, R. On a class of equations reducible to Laplace's equation. Ganita **5** (1954), 93–96 (1955).

The author investigates the most general type of second order partial differential equations in two and three dimensions which can be reduced to Laplace's equation by, in polar coordinates, changing the polar coordinate $\rho=\rho(r)$ without altering the angular coordinates. *L. Nirenberg* (New York, N.Y.).

Pini, Bruno. Sull'unicità della soluzione del problema di Dirichlet per le equazioni lineari ellittiche in due variabili. Rend. Sem. Mat. Univ. Padova **26** (1956), 223–231.

Using well-known L_2 inequalities for elliptic differential equations it is not difficult to show that a solution of the Dirichlet problem for a strongly elliptic system is unique provided the domain has sufficiently small diameter. (References are given in this paper.) The author gives a straightforward proof of the uniqueness for a single equation in the plane, where the L_2 inequalities are particularly simple to derive. *L. Nirenberg.*

Amerio, Luigi. Teoremi di esistenza per i problemi di Dirichlet e di Neumann per l'equazione $\Delta_2 u-ku=0$. Ricerche Mat. **5** (1956), 58–96.

Let τ be a finite plane domain whose boundary σ consists of a finite number of simple, closed, non-intersecting curves of class C^2. Let ν_M denote the inner normal to σ at the point M. A solution in τ of $\Delta u-ku=0$ (k real or complex) is said to belong to class E, if, as P tends to M along ν_M, $u(P)$ and $\partial u(p)/\partial\nu_M$ have limits $A(M)$ and $B(M)$, respectively, almost everywhere on σ, and if

$$u(P)=\frac{1}{2\pi}\int_\sigma\Big\{A(M)\frac{\partial v(P,M)}{\partial\nu_M}-B(M)v(P, M)d\sigma_M, \text{ if } P\in\tau,$$

$$0=\int_\sigma\Big\{A(M)\frac{\partial v(Q, M)}{\partial\nu_M}-B(M)v(Q, M)\Big\}d\sigma_M, \text{ if } Q\notin\tau+\sigma,$$

where v is the fundamental solution of the equation. An analogous class E' is defined of solutions of the equation in an infinite region τ'. In this paper, necessary and sufficient conditions are given for the existence of solutions of class E or E' in τ or τ', respectively, of Neumann's and Dirichlet's problems. Neumann's problem in τ: If k is not an eigenvalue, one and only one solution of class E corresponds to any assigned, summable $B(M)$. If k is an eigenvalue, a solution of class E exists, if, and only if, $B(M)$ is orthogonal on σ to the eigensolutions of the problem. Dirichlet's problem in τ: A necessary condition that there exist a solution in E corresponding to given $A(M)$ is that the double layer $\int_\sigma A(M)\partial v(P, M)/\partial\nu_M\,d\sigma_M$ belong to E. If k is not an eigenvalue, this condition is also sufficient for the existence of a solution in E (which, in this case, must be unique). If k is an eigenvalue, this condition and the additional requirement that $A(M)$ be orthogonal on σ to the normal derivatives of the eigensolutions of the problem are both necessary and sufficient. *A. Douglis* (New York, N.Y.).

Boigelot, A. Sur la solution du problème aux limites de Dirichlet-Neumann, relatif à l'opérateur des ondes. Bull. Soc. Roy. Sci. Liège **25** (1956), 405–413.

L'auteur montre l'existence et l'unicité de la solution du problème de Dirichlet-Neumann (i.e. de Dirichlet sur une partie de la frontière et de Neumann sur le reste de cell-ci) posé dans l'ouvert $\Omega\times]0, \infty[$, Ω: ouvert arbitraire de R^n, pour l'opérateur

$$-\Delta+a\frac{\partial^2}{\partial t^2}+b\frac{\partial}{\partial t}+c\ (a>0),$$

les données initiales étant prises uniformément dans tout compact de Ω. Il suppose que les données initiales u_0 et u_1 ainsi que la donnée au second membre $m(x, t)$ (pour tout $t\in]0, \infty[$ vérifient les conditions de Dirichlet-Neumann en même temps que leurs laplaciens itérés un nombre arbitraire de fois. Il montre que la solution existe, est unique et vérifie les conditions imposées à la donnée au second membre. De plus, cette solution permet d'introduire un opérateur matriciel qui obéit à une loi de semi-groupe, pour autant que le second membre soit nul. *H. G. Garnir* (Liège).

Greco, Donato. Il problema di derivata obliqua per certi sistemi di equazioni a derivate parziali di tipo ellittico in due variabili. Ann. Mat. Pura Appl. (4) **42** (1956), 1–24.

Siano $\mathfrak{M}^{(h)}[u]$ $(h=1, 2, \cdots, m)$ m operatori lineari ellittici del secondo ordine in due variabili. L'A. considera il sistema di equazioni a derivate parziali $\mathfrak{M}^{(h)}[u_h]=\infty$ nelle funzioni incognite $u_1, u_2, \cdots, u_m$, associando ad esso le condizioni al contorno

$$\sum_{k=1}^m\Big[\varphi_{hk}\frac{d\varphi_k}{d\nu_k}+\alpha_{hk}\frac{du_k}{ds}+\beta_{hk}u_k\Big]=f_h\ (h=1, 2, \cdots, m),$$

dove ν_k è la conormale relativa all'operatore $\mathfrak{M}^{(k)}$, s è l'ascissa curvilinea sulla frontiera di un dominio T, anche a più contorni, e $q_{hk}, \alpha_{hk}, \beta_{hk}, f_h$ sono funzioni di s. Sotto opportune ipotesi di regolarità per tali funzioni, per i coefficienti degli operatori $\mathfrak{M}^{(h)}$ e per il dominio T, e rappresentando le funzioni incognite mediante potenziali generalizzati di semplice strato, tale problema al contorno si lascia tradurre in un sistema di equazioni integrali a valor principale, il cui indice può essere calculato in termini finiti. Nel caso regolare, quando cioè riesca: det $\|q_{hk}\|\neq0$, l'A. indica alcuni casi in cui il problema ha carattere fredholmiano ed altri in cui sussiste per esso il teorema di esistenza e di unicità della soluzione. Nel caso generale le eventuali condizioni di compatibilità sono caratterizzate collegandole con le soluzioni di un certo problema aggiunto omogeneo. I risultati ottenuti nello studio di questo problema aggiunto sono da considerarsi del tutto nuovi anche per il caso di una sola equazione. *C. Miranda* (Napoli).

Browder, Felix E. Regularity theorems for solutions of partial differential equations with variable coefficients. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 234–236.

This paper gives a further generalization of the famous Weyl lemma that weak solutions of elliptic partial differential equations are strong solutions. More specifically the author formulates a nice generalization, from the case of constant coefficients to that of sufficiently differ-

entiable coefficients, of the theorem of Hörmander [Acta Math. **94** (1955), 161–248; MR **17**, 853] giving the desired result from a single algebraic necessary and sufficient condition on the coefficients, which includes both the elliptic and the parabolic cases. The proof also follows Hörmander's argument, using Fourier transforms of Schwartzian distributions. *F. H. Brownell.*

Skorobogat'ko, V. Ya. Theorems in the qualitative theory of partial second order differential equations. Ukrain. Mat. Ž. 8 (1956), 435–440. (Russian)

Theorems analogous to certain of those for ordinary differential equations are proved for the equation

$$(*)\qquad \frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+c(x_1, x_2)u=0 \quad (c\geqq 0).$$

Among the results are the following. i) Let solution u of (*) be such that $u>0$ in a region D_1 with piecewise smooth boundary τ_1 and such that $u/\tau_1=0$. (Boundary τ_1 is said to be a "nodal line" of u.) Then the "inner diameter" d_1 of D_1 satisfies the inequalities: $\pi/\sqrt{M}\leqq d_1\leqq 2\mu_0/\sqrt{m}$, where $M=\max_{x\in D_1} c$ and $m=\min_{x\in D_1} c$ and μ_0 is the first zero of the Bessel function $I_0(x)$ of zeroth order. ii) Besides the hypotheses of i), assume c is a positive constant and that the set of points x in τ_1 such that there is no circle in D_1 having x on its boundary has linear measure zero. If Z is a solution in D_1 of (*), and if Z and solution u are linearly independent, then solution z is zero at some point of D_1. iii) Given

$$(**)\qquad \frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+c(x_1, x_2, \lambda)u=0$$

in region D such that for all $(x_1, x_2)\in D$,

$$\lim_{\lambda\to\infty} c(x_1, x_2, \lambda)=\infty.$$

Then if region D_ν is such that $\bar{D}_\nu\subset D$, there exists λ_ν such that for each $\lambda\geqq\lambda_\nu$, the nodal line of a solution u of (**) has a non-empty intersection with D_ν. *J. Cronin.*

Vekilov, Š. I. Linear and non-linear mixed boundary problems for a set of harmonic functions. Akad. Nauk Azerbaĭdžan. SSR. Trudy Inst. Fiz. Mat. **4-5** (1952), 128–148. (Russian. Azerbaijanian summary)

The author considers both linear and non-linear problems of the form such that $U_1(M)$, $U_2(M)$, $\cdots$, $U_N(M)$ are harmonic for $M\in$ a connected region D, with the boundary conditions: (Linear case)

$$\frac{dU_k}{dn}=\sum_{j=1}^{N} a_{kj}(M)U_j(M)+f_k(M),$$

($M\in$ boundary S of D); or else (Non-linear case)

$$(*)\qquad \frac{dU_k}{dn}+\sum_{j=1}^{N} a_{kj}(M)U_j(M)=\lambda_k\Phi_k\Big(M, U_1, U_2, \cdots, U_N, \frac{dU_1}{dn}, \cdots, \frac{dU_N}{dn}\Big).$$

The linear case is solved by reduction to Fredholm integral equations whose spectrum can be studied; this is carried through for both two and three dimensions.

The non-linear case is solved (for two dimensions) by invoking a fixed-point theorem for an operator which diminishes distance. The operator is set up by introducing a set of densities $\boldsymbol{\mu}=(\mu_1, \mu_2, \cdots, \mu_N)$ for single-layer distributions on the boundary S. The densities $\boldsymbol{\mu}$ determine potentials $\mathbf{U}=(U_1, U_2, \cdots, U_N)$; introducing $\mathbf{U}$ into the Φ_k produces functions of the form $f_k(M)$. The equations (*) can now be cast as linear equations for a new vector $\mathbf{U}^*$:

$$\frac{dU_k^*}{dn}+\sum_{j=1}^{N} a_{kj}(M)U_j^*(M)=\lambda_k f_k(M)$$

and hence solved by the previously-developed linear methods. The U_k^* determine new single-layer densities $\boldsymbol{\mu}^*=(\mu_1^*, \mu_2^*, \cdots, \mu_N^*)$. This determination of $\boldsymbol{\mu}^*$ directly from $\mathbf{U}^*$ would of course involve integral equations of the first kind; but if, instead, one substitutes into the linear form of (*) and uses the jump condition for the normal derivative at a single layer, one obtains Fredholm equations

$$\mu_k^*(M)=\int_s \sum H_{kj}(M, R)\mu_j^*(R)ds_R+f_k(M, \boldsymbol{\mu}).$$

It can now be shown (with the anticipated Lipschitz condition on the Φ_k) that, for sufficiently small λ_k, $d(\mu^*, w^*)<ad(\boldsymbol{\mu}, \mathbf{w})$, for $0<a<1$, where the metric is defined by

$$d(\boldsymbol{\mu},\mathbf{w})\equiv\sum_{k=1}^{N} \max_{M\in S} |\mu_k(S)-w_k(S)|,$$

and the corresponding space is complete.

R. B. Davis (Syracuse, N.Y.).

Brownell, F. H. Extended asymptotic eigenvalue distributions for bounded domains in n-space. J. Math. Mech. **6** (1957), 119–166.

The author considers the asymptotic behavior for large λ of the function $N(\lambda)$, which is defined as the number of eigenvalues λ_j less than λ of either the free or the fixed membrane problem in an n-dimensional domain D. The asymptotic series in powers of ω^{-1} of $\sum \lambda_j^{-1}(\lambda_j+\omega^2)^{-1}$ has been given by A. Pleijel [Ark. Mat. **2** (1952), 553–569; MR **15**, 798, 1140] under the assumption that the boundary of D is infinitely differentiable. This result was used by the author in a previous paper [Pacific J. Math. **5** (1955), 483–499; MR **18**, 216] to estimate $N(\lambda)$ for large λ, where not the order of the error, but only of the Gaussian average of the error term over large values of λ is given.

In the present paper the author derives the analogues of Pleijel's and his own results for several larger classes of domains. In particular the class of n-dimensional domains with twice Hölder-continuously differentiable boundary is treated.

The author also treats two-dimensional domains bounded by finitely many Jordan arcs. In this case, the asymptotic form contains a term depending explicitly on the angles of corners. *H. F. Weinberger.*

Ficken, F. A. A derivation of the equation for a vibrating string. Amer. Math. Monthly **64** (1957), 155–157.

Hellwig, Günter. Über die Anwendung der Laplace-Transformation auf Randwertprobleme. Math. Z. **66** (1957), 371–388.

Set

$$\bar{R}_t u=a_{11}u(0, t)+a_{12}u_x(0, t)+a_{13}u_t(0, t)+b_{11}u(1, t)+b_{12}u_x(1, t)+b_{12}u_t(1, t),$$

$$R_t v=(a_{11}+sa_{13})v(0, s)+a_{12}v_x(0, s)+(b_{11}+sb_{13})v(1, s)+b_{12}v_x(1, s).$$

The a_{11}, $\cdots$, b_{13} are constants. For the constants associated with u_t in $\bar{R}_t u$ it is assumed that $a_{13}b_{23}-a_{23}b_{13}\neq 0$. Use is

made of the Laplace transformation

$$v(x,s)=Lu=\int_0^\infty e^{-st}u(x,t)dt$$

to solve the following boundary value problem for the heat equation

$$(*)\quad \begin{aligned} &-u_{xx}+q(x)u+u_t=0 && (0<x<1,\ 0<t<\infty),\\ &\lim_{t\to 0} u(x,t)=u_0(x) && (0<x<1),\\ &\bar{R}_t u=0, && (0<t<\infty). \end{aligned}$$

The following system is set up for $v=v(x,s)$ to satisfy

$$(**)\quad -\frac{d^2v}{dx^2}+qv+sv=u_0(x),\ R_tv=0.$$

Making use of the appropriate Green's function $G(x,y,s)$ the solution of (**) can be written in the form $v_1(x,s)=\int_0^1 G(x,y,s)u_0(y)dy$. After making a careful analysis of the asymptotic properties of $G(x,y,s)$ the author proves that if $g(x)$, $u_0'''(x)$ are continuous on $0\leqq x\leqq 1$ and if $g(x)$ is of bounded variation, then the inverse of v_1, namely

$$u(x,t)=L^{-1}v_1=(2\pi i)^{-1}\ \text{H.W.}\int_{\xi-i\infty}^{\xi+i\infty} e^{st}v_1(x,s)ds,$$

is a solution of system (*). It is to be noted that the application of the Laplace transformation to system (*) may not yield system (**). This situation is permissible since no restrictions were placed on $u(x,t)$ at the points (0, 0) and (1, 0). *F. G. Dressel* (Durham, N.C.).

Thyssen, M. Solution élémentaire d'un opérateur hyperbolique décomposable du quatrième ordre. Bull. Soc. Roy. Sci. Liège **25** (1956), 371–382.

Le reviewer a donné [Acad. Roy. Belg. Bull. Cl. Sci. (5) **38** (1952), 1129–1141; MR **14**, 1090] l'expression d'une solution élémentaire de l'opérateur

$$(-\Delta+k^2)(-\Delta_a+l^2),\ \Delta=\sum_{k=1}^n \frac{\partial^2}{\partial x_k^2},\ \Delta_a=\sum_{k=1}^n a_k^2\frac{\partial^2}{\partial x_k^2},$$

qui ne fait intervenir qu'une seule intégration portant sur la solution élémentaire (classique) de l'opérateur

$$(-\Delta+k^2)^2.$$

En recourant à une transformation de Laplace inverse, l'auteur en déduit une formule analogue donnant la solution élémentaire de l'opérateur hyperbolique

$$\left(-\Delta+\frac{\partial^2}{\partial t^2}+\alpha_1\frac{\partial}{\partial t}+\alpha_2\right)\left(-\Delta+\frac{\partial^2}{\partial t^2}+\beta_2\frac{\partial}{\partial t}+\beta_2\right),$$

à partir de celle de l'opérateur

$$\left(-\Delta+a\frac{\partial^2}{\partial t^2}+b\frac{\partial}{\partial t}+c\right)^2.$$

Pour $n=1$, 2 ou 3, il discute avec soin le support de la solution élémentaire obtenue, dans le but de vérifier explicitement que ce support est l'enveloppe connexe des supports des solutions élémentaires de chacun des facteurs de l'opérateur étudie. Lorsque $\alpha_1=\alpha_2=\beta_1=\beta_2=0$, l'auteur retrouve très simplement les résultats de G. Herglotz [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. **80** (1928), 69–114] et de F. Bureau [Acad. Roy. Belg. Bull. Cl. Sci. **34** (1948), 566–592; Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8°, **21** (1948), no. 6; MR **10**, 459; **11**, 253]. *H. G. Garnir* (Liège).

Èl'sgol'c, L. È. On the integration of linear partial differential equations with retarded argument. Moskov. Gos. Univ. Uč. Zap. **181**. Mat. **8** (1956), 57–58. (Russian)

The author discusses the possibility of applying the method of separation of variables to the solution of partial differential equations in which the time variable is retarded. *R. Bellman* (Santa Monica, Calif.).

Mitrinovitch, D. S. Sur une équation linéaire aux dérivées partielles à coefficients constants. Math. Gaz. **41** (1957), 41–43.

Certain elementary facts, of pedagogical interest and apparently not published elsewhere, about the relations between the solutions of a partial differential equation and the nature of the roots of its characteristic equation.

Dobryšman, E. M.; and Dyubyuk, A. F. On the solution of the equation $\left(\frac{\partial^2}{\partial t^2}-\frac{\partial}{\partial t}\Delta-\Delta\right)u=f$. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 55–58. (Russian)

Using the methods of the operational calculus, the authors obtain an explicit solution of the Cauchy problem:

$$\begin{gathered}\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial t}\Delta u-\Delta u=f(x,y,z,t),\\ u(x,y,z,0)=u_0(x,y,z),\\ \frac{\partial}{\partial t}u(x,y,z,0)=u_1(x,y,z).\end{gathered}$$

The authors also find a solution defined for $z>0$ and satisfying the additional condition $\partial u/\partial z=0$ when $z=0$. They state that the same method applies to the case $u\alpha+\beta\partial u/\partial z=R(x,y,t)$ when $z=0$. *R. Finn.*

Temlyakov, A. A. Analytic solutions of the wave equation in two-dimensional space. Moskov. Oblast. Pedagog. Inst. Uč. Zap. Trudy Kafedr Mat. **20** (1954), 17–36. (Russian)

The boundary value problem considered is the determination of a solution $F(x,y,z)$ (analytic in (x,y,z)) of the wave equation

$$\frac{\partial^2F}{\partial x^2}+\frac{\partial^2F}{\partial y^2}-\frac{\partial^2F}{\partial z^2}=f(x,y,z),$$

where the function $f(x,y,z)$ is supposed to be analytic in (x,y,z) at (0, 0, 0), and the function $F(x,y,z)$ takes on prescribed values on the characteristic planes $z=y$ and $z=-y$; that is, $F(x,y,y)=\varphi(x,y)$ and $F(x,y,-y)=\psi(x,y)$, with $\varphi(x,y)$ and $\psi(x,y)$ given functions, analytic in (x,y) at (0, 0) and satisfying $\varphi(x,0)=\psi(x,0)$.

This problem is the (x,y,z)-analogue of the Goursat problem:

$$\frac{\partial^2F}{\partial x^2}-\frac{\partial^2F}{\partial z^2}=f(x,z),\ F(x,x)=\varphi(x),\ F(x,-x)=\psi(x),\quad \varphi(0)=\psi(0).$$

An explicit solution is given, based on a previous result of the author [Mat. Sb. N.S. **5**(47) (1939), 487–495; MR **1**, 311] and a lemma which asserts the expansion of an arbitrary function $f(x,y,z)$, analytic in (x,y,z) at (0, 0, 0), in a series of the form:

$$f(x,y,z)=\sum_{n=0}^\infty v_n(x,y,z)\frac{(z-y)^n}{n!}.$$

J. B. Diaz (College Park, Md.).

Temlyakov, A. A., and Sohan', A. M. Analytic solution of a certain problem of Goursat. Moskov. Oblast. Pedagog. Inst. Uč. Zap. Trudy Kafedr Mat. 21 (1954), 23–33. (Russian)

The "Goursat" boundary value problem solved is that of determining a solution $F(x, y, z)$ (analytic in (x, y, z)) of the partial differential equation

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}-\frac{\partial^2 F}{\partial z^2}+a\frac{\partial F}{\partial x}+b\frac{\partial F}{\partial y}+c\frac{\partial F}{\partial z}+fF=0,$$

(where a, b, c, f are constants) satisfying the following conditions on the characteristic planes $z=y$ and $z=-y$:

$$F(x, y, y)=y\varphi(x, y), \quad F(x, y, -y)=y\psi(x, y),$$

where φ and ψ are given functions, analytic in (x, y) at $(0, 0)$. *J. B. Diaz* (College Park, Md.).

Maurin, K. Über gemischte Rand- und Anfangswertprobleme im Grossen für gewisse Systeme von Differentialgleichungen auf differenzierbaren Mannigfaltigkeiten. (Eine Begründung der Fourierschen Methode.) Studia Math. 15 (1956), 314–327.

Let A be an elliptic system of formally self-adjoint differential operators of σ order with sufficiently smooth complex coefficients in an n-dimensional differentiable manifold Ω. Let $C_0^{\infty,r}(\Omega)$ be the space of infinitely differentiable r-vectors, r denoting the order of the system, and let $L^{2,r}(\Omega)$ be the Hilbert space obtained as the completion of $C_0^{\infty,r}(\Omega)$ with respect to the norm

$$(u, v)=\sum_{i=1}^{r}\int_\Omega u_i(x)\overline{v_i(x)}dx.$$

Consider A as an additive operator A_0 on $D(A_0)=C_0^{\infty,r}(\Omega)$ into $L^{2,r}(\Omega)$. It is assumed that $-A_0$ is semi-bounded from below and that $-A_0$ has a self-adjoint extension $-A_1$ with the same lower bound α: $-A_1=\int_\alpha^\infty \lambda dE(\lambda)$. The principal theorem states that, if $f\in D(A_1^{m+1})$, $g\in D(A_1^m)$ with $m>n/2\sigma+1$, then

$$u(x, t)=\int_\alpha^\infty \cos\lambda^{\frac12}t dE(\lambda)f(x)+\int_\alpha^\infty \lambda^{-\frac12}\sin\lambda^{\frac12}t dE(\lambda)g(x)$$

is equivalent to a vector $\tilde u(x, t)$ which is a genuine solution of the problem $\partial^2\tilde u/\partial t^2=A\tilde u$, $\tilde u(x, 0)=f(x)$, $\tilde u_t(x, 0)=g(x)$; it is proved that this problem is correctly set in the sense of Hadamard. Parabolic and quantum-mechanical equations also are treated similarly. *K. Yosida.*

Payne, L. E.; Pólya, G.; and Weinberger, H. F. On the ratio of consecutive eigenvalues. J. Math. Phys. 35 (1956), 289–298.

The eigenvalues considered are those in the three problems for a two-dimensional domain D with boundary C, and

(1) $\Delta u+\lambda u=0$ in D, $u=0$ on C,

(2) $\Delta\Delta u-\mu u=0$ in D, $u=\partial u/\partial n=0$ on C,

(3) $\Delta\Delta u+\nu\Delta u=0$ in D, $u=\partial u/\partial n=0$ on C.

Let λ_n, μ_n, ν_n be the eigenvalues in these problems respectively. It is proved that

$$\lambda_{n+1}\leq\lambda_n+2(\lambda_1+\cdots+\lambda_n)/n,$$
$$\mu_{n+1}\leq\mu_n+8(\mu_1+\cdots+\mu_n)/n,$$

so that $\lambda_{n+1}\leq 3\lambda_n$, $\mu_{n+1}\leq 9\mu_n$. The method (selection of appropriate trial-functions) is not so successful for (3), in which it is proved only that $\nu_2<3\nu_1$. A proof of $\lambda_2+\lambda_3\leq 6\lambda_1$ is also sketched.

E. C. Titchmarsh (Oxford).

Campanato, Sergio. Teoremi di completezza relativi al sistema di equazioni dell'equilibrio elastico. Rend. Sem. Mat. Univ. Padova 25 (1956), 122–137.

Let $\mathbf{u}$ be a vector field with two components (u_1, u_2) defined in the domain D of the xy-plane, bounded by a single twice differentiable curve $\mathscr{F}D$. The author considers the class S of the solutions of the differential equation

$$\Delta_2\mathbf{u}+k\operatorname{grad}\operatorname{div}\mathbf{u}=0$$

with continuous first derivatives in $D+\mathscr{F}D$, where k is a non-negative constant.

Let $\mathscr{F}D_1$ and $\mathscr{F}_2D$ be two disjoint arcs such that $\mathscr{F}D=\mathscr{F}_1D+\mathscr{F}_2D$. Let S_0 denote the sub-class of S constituted by vectors $\mathbf{u}$ satisfying on $\mathscr{F}_2D$ the boundary condition:

$$k\operatorname{div}\mathbf{u}\cdot n_1+\frac{du_1}{dn}+\frac{du_2}{ds}=0,$$

$$k\operatorname{div}\mathbf{u}\cdot n_2+\frac{du_2}{dn}-\frac{du_1}{ds}=0.$$

($\mathbf{n}\equiv(n_1, n_2)$ is the inward normal to $\mathscr{F}D$).

Let $\mathscr{H}$ be the Hilbert space of vectors $v\equiv(v_1, v_2)$ defined on $\mathscr{F}_1D$, with the following scalar product

$$(v,v')=\sum_{i=1}^{2}\int_{\mathscr{F}_1D}v_iv_i'ds.$$

The author proves that S_0 is a base for $\mathscr{H}$.

The result is of main interest in the mathematical theory of plane elasticity. *G. Fichera* (Rome).

See also: Prodi, p. 877; Zahorska, p. 885; Maslov, p. 916; Mikeladze, p. 938; Babuška, p. 938; Haselgrove and Hoyle, p. 939; Hida, p. 945; Kampé de Fériet, p. 949; Jones, p. 967; Scott, p. 968; Gasanov, p. 971; Whitesitt, p. 869; Fourès-Bruhat, p. 976.

Difference Equations, Functional Equations

Čermák, Jiří. Bemerkung zum Grenzübergange von Differenzengleichungen in Differentialgleichungen. Časopis Pěst. Mat. 81 (1956), 224–228. (Czech. Russian and German summaries)

Das System von Differenzengleichungen

$$\underset{\omega}{\Delta}u_i(x)=\sum_{j=1}^{n}a_{ij}u_j(x), \left(\underset{\omega}{\Delta}u_i(x)=\frac{u_i(x+\omega)-u_i(x)}{\omega}\right),$$
$$(i=1, 2, \cdots, n),$$

kann in Matrizenbezeichnung durch

$$(*)\qquad \underset{\omega}{\Delta}\mathbf{u}(x)=A\mathbf{u}(x)$$

abgekürzt werden. Die Matrix A besteht dabei aus den komplexen konstanten Zahlen a_{ij}. Mit $\omega\to 0$ entsteht aus (*) die Matrizendifferentialgleichung

$$\frac{d\mathbf{u}}{dx}=A\mathbf{u}(x),$$

also ein System von n linearen homogenen Differentialgleichungen erster Ordnung mit konstanten Koeffizienten für die n unbelkannten Funktionen $u_i(x)$. Zu jedem Eigenwert a der Matrix A gehört dann eine Schar von Lösungen der Differenzengleichungen, die aus sovielen Lösungen besteht, wie die Vielfachheit dieses Eigenwertes angibt.

Dabei erscheinen diese Lösungsvektoren $\mathfrak{u}^{\mu\nu}$ als gewisse Linearkombinationen der Eigenvektoren $\mathfrak{a}^{\mu\nu}$ der Matrix A. Durch den Grenzübergang $\omega\to 0$ erhält Verfasser in Übereinstimmung mit gleichartigen Ergebnissen von O. Borůvka und A Walther (zur Integrationstheorie linearer Differentialsysteme mit konstanten Koeffizienten) aus der Lösungsschar des Differenzensystems die Lösungsschar

$$\mathfrak{u}_{\mu\nu}=e^{ax}\left\{\mathfrak{a}_{\mu\nu}+\frac{x}{1!}\mathfrak{a}_{\mu+1,\nu}+\frac{x^2}{2!}\mathfrak{a}_{\mu+2,\nu}+\cdots+\frac{x^{r-\mu}}{(r-\mu)}\mathfrak{a}_{r\nu}\right\}$$
$$(1\leq\mu\leq r,\ \nu=1,2,\cdots,\alpha_{r-\mu+1})$$

des Differentialsystems (*). Dabei sind die $\alpha_1, \alpha_2, \cdots, \alpha_{r-\mu+1}$ E. Weyr's charakteristische Zahlen, welche den Eigenwerte a der Matrix A entsprechen [cf. O. Borůvka, Časopis Pest. Mat. **79** (1954), 151–155; MR **16**, 475; A. Walther, Math. Ann. **95** (1955), 257–266]. *M. Pinl.*

Boas, R. P., Jr. Functions which are odd about several points: addendum. Nieuw Arch. Wisk. (3) **5** (1957), 25.

The author states that in his original paper [same Arch. (3) **1** (1953), 27–32; MR **14**, 987] he proved the following lemma: If a measurable function f satisfies $f(t+a)=f(t)$ for almost every t and for each a of a set having 0 as a limit point then $f(t)$ is a constant for almost all t. This lemma was also proved by C. Burstin [Monatsh. Math. Phys. **26** (1915), 229–262] and others. On the other hand, one of the proofs the author gave for this lemma contained several misprints and is reproduced here in a corrected form. *J. Aczél* (Debrecen).

Sobol', I. M. Positive solutions of linear differential equations with retardation. Moskov. Gos. Univ. Uč. Zap. **181.** Mat. **8** (1956), 45–56. (Russian)

The author shows that the functional equation $u''(x)=m(x)u(x-d(x))$ in the case where $m(x)\geq 0$ and $d(x)\geq 0$, possesses solutions which act like 1 and x asymptotically if the integral $\int^\infty xm(x)dx$ is convergent, and solutions act like exponentials with positive and negative exponent if the integral diverges. *R. Bellman.*

Popov, B. S. Sur la résolution générale d'une classe d'équations. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 1107–1109.

For equations of the form $\sum\{f(x)\}^{p_i}=A$, written also as $\sum e^{p_i u}=A$, a formula is given for the error made on taking $u=u_0$ as solution. The derivation, generalization and application of the formula are promised to appear in a subsequent paper. *M. Marden* (Milwaukee, Wis.).

See also: Doetsch, p. 894; Vvedenskaya, p. 901; Tihonov and Samarskiĭ, p. 938; Kopal and Kurth, p. 977.

Integral and Integrodifferential Equations

Sofronov, I. D. On certain properties of singular operators and solutions of singular integral equations. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 940–942. (Russian)

The author considers integral operators $f=K\varphi$ defined by equations of the form

$$(1)\quad f(x)=a(x)\varphi(x)+\frac{1}{2\pi}\int_0^{2\pi}n(t,x)\cot\tfrac{1}{2}(t-x)\varphi(t)dt,$$

where $a(x)$ and $n(t,x)$ satisfy a Hölder condition of order α, and $0<\alpha\leq 1$. He states (i) that if φ satisfies a Hölder condition of order α, then $f(x)$ satisfies a Hölder condition of order β for all $\beta<\alpha$, and (ii) that if (1) has a unique solution φ for all f, and f satisfies a Hölder condition of order α, then φ satisfies a Hölder condition of order β for all $\beta<\alpha$. Explicit inequalities for the Hölder constants are given. In the particular case

$$f(x)=a(x)\varphi(x)+\frac{n(x,x)}{2\pi}\int_0^{2\pi}\cot\tfrac{1}{2}(t-x)\varphi(t)dt,$$

one can take $\beta=\alpha$. Relations between the differentiability properties of $f(x)$ and $\varphi(x)$ are also stated. No proofs are given. *F. Smithies* (Cambridge, England).

Ivanov, V. V. An approximate solution of singular integral equations. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 15–18. (Russian)

Let X and Y be Hilbert spaces (not necessarily complete) and let $x_0, x_1, x_2, \cdots, x_n, \cdots$ be a complete orthonormal sequence of elements in the space X. Suppose that G and T are linear operators on X with values in Y, and that the inverse of G exists, while T is completely continuous. The author sketches the proof of the following theorem: if the operator $G+\lambda T$ has an inverse, where λ is a complex number, and y is an element of Y, then for all n sufficiently large, the system of linear equations

$$\sum_{k=0}^{n}(Gx_k+\lambda Tx_k, Gx_j)\beta_k=(y, Gx_j)\ (j=0,1,\cdots,n)$$

has a unique solution $\beta_{0n}, \beta_{1n}, \beta_{2n}, \cdots, \beta_{nn}$, and the sequence of vectors $\sum_{k=0}^{n}\beta_{kn}x_k$ $(n=0, 1, \cdots)$ converges strongly to an element x of X such that $Gx+\lambda Tx=y$. This result is then applied to the approximate solution of the singular integral equation

$$A(t_0)x(t_0)+\frac{B(t_0)}{\pi i}\int_\gamma\frac{x(t)dt}{t-t_0}+\lambda\int_\gamma T(t,t_0)x(t)dt=y(t_0),$$

where γ is the unit circle with center at the origin, the functions y, A, B, T are defined on γ and belong to the Hölder class $H(\mu)$ with exponent μ, $0<\mu<1$, and $A^2-B^2\neq 0$ on γ. *J. B. Diaz* (College Park, Md.).

MacNerney, J. S. Determinants of harmonic matrices. Proc. Amer. Math. Soc. **7** (1956), 1044–1046.

Suppose F is an $n\times n$ matrix of complex-valued functions from the real numbers, continuous and of bounded variation on every interval, such that $F(0)=0$ and M the harmonic matrix [H. S. Wall, Arch. Math. **5** (1954), 160–167; MR **15**, 801] such that

$$M(s,t)=I+\int_s^t dF(u)\cdot M(u,t).$$

The author establishes the formula:

$$\det M(s,t)=\exp\left(\sum_1^n[F_{pp}(t)-F_{pp}(s)]\right)$$

and an analogous formula for the determinant of his quasi-harmonic matrix [J. S. MacNerney, J. Elisha Mitchell Sci. Soc. **71** (1955), 185–200; MR **18**, 54]. *H. S. Wall* (Austin, Texas).

See also: Doetsch, p. 894; Vekilov, p. 903; Èskin, p. 912; Ladyženskiĭ, p. 914.

Calculus of Variations

See: Reid, p. 898; Ficken, p. 903; Kahan, Rideau and Roussopoulos, p. 975; Osborn, p. 980.

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

★**Montgomery, Deane. Topological transformation groups.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 185–188. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

A short exposition of the facts about locally compact groups and certain pathological examples of the action of compact groups on manifolds. There is a preliminary announcement of results, since published, obtained by the author and Samelson on the fixed-points of an orientation preserving simplicial homeomorphism of the three-sphere of period two [Canad. J. Math. 7, (1955), 208–220; MR 16, 946]. *A. M. Gleason* (Cambridge, Mass.).

Ganea, T. Aspects of the theory of topological groups. Gaz. Mat. Fiz. Ser. A. 8 (1956), 510–518. (Romanian)

Expository article with especial attention to Hilbert's fifth problem.

Vilenkin, N. Ya. On a class of locally compact zero-dimensional topological groups. Mat. Sb. N.S. **40(82)** (1956), 479–496. (Russian)

This paper is devoted to the proof of theorems that dualize those proved in earlier papers [cf. Mat. Sb. N.S. 33(75) (1953), 37–44; MR 15, 399].

Among the preliminary results we mention the theorem that the character group of a fully properly stratified group is also fully properly stratified. *W. T. van Est.*

Melencov, A. A. Cut sets in connected topological groups. Ukrain. Mat. Ž. 8 (1956), 289–298. (Russian)

The closed subgroup H of the connected topological group G is called a cutset if it is a cutting (i.e. $G-H$ is not connected, and dense in G), and a simple cutset if, moreover, $(G-H)\cup N$ is connected and H/N discrete (here N is the e-component of H). Some of the results: 1. A simple cutset in a locally connected group is irreducible (no proper subset separates G); 2. with the same hypotheses, write $G-H=P_1\cup P_2$ with $\bar{P}_1\cap\bar{P}_2\cap(G-H)=\emptyset$; then $P_1=P_2^{-1}$, P_1 and P_2 are open connected, the decomposition is unique, $\bar{P}_1\cap\bar{P}_2=H$; 3. if moreover $H=N$, then P_1 and $P_1\cup H$ are semigroups, all elements of finite order belonging to H. *H. Samelson.*

Braconnier, Jean. L'analyse harmonique dans les groupes abéliens. I. Enseignement Math. (2) 2 (1956), 12–41.

An exposition, with brief indications of the ideas of proofs, of the modern theory of harmonic analysis. The author states that the second part shall be devoted to the Fourier-Laplace transform, the spectral theory of continuous functions and the relation between the group representation and the Fourier transform. *K. Yosida.*

Maurer, I. Eine Topologisierung der Gruppe unendlicher Permutationen. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 8 (1956), 265–272. (Romanian. Russian and German summaries)

An infinite permutation $\pi(n)$ means a one-one mapping of the set of natural numbers onto itself. A sequence $\pi_r(n)$ of infinite permutations has limit $\pi(n)$ if for any natural number n there exists a number R_n such that $\pi_r(n)=\pi(n)$ whenever $r>R_n$. {This definition of limit, ascribed by the author to L. Onofri, Ann. Mat. Pura Appl. (4) 4 (1927), 73–106, was in fact given by G. Andreaoli, Rend. Circ. Mat. Palermo 40 (1915), 299–335.} On the basis of this definition the author introduces in the usual manner the concept of closure into the space S_∞ of infinite permutations, and shows that it is then a topological space in the sense of Riesz-Kuratowski [C. Kuratowski, Topologie, I, 2nd ed., Warszawa-Wrocław, 1948; MR **10**, 389]. It is shown that any permutation is the limit of a sequence of finite permutations, so that the set S of finite permutations has closure S_∞. S_∞ is also a topological group with respect to the obvious group operation; and the author applies work of Schreier and Ulam [Studia Math. 4 (1933), 134–141; Fund. Math. 28 (1936), 258–260] to show that this topological group is simple and has only inner automorphisms. *I. M. H. Etherington.*

Koch, R. J.; and Wallace, A. D. Stability in semigroups. Duke Math. J. 24 (1957), 193–195.

A semigroup S is said to be stable if $a, b\in S$ and $SaCSab$ imply $Sa=Sab$, and if $aSCbaS$ implies $aS=baS$. The authors improve a theorem of J. A. Green [Ann. of Math. (2) **54** (1951), 163–172; MR 13, 100] by showing that the conclusions of his Theorem 8 hold if S is merely assumed to be stable. They also show that if S is a topological semigroup then any one of the following conditions will imply stability: S is commutative, S is compact, S is the union of groups, $\{x^n\}$ has compact closure for each $x\in S$. The original theorem of Green did not apply, for example, to the ordinary closed unit interval.
A. Shields (Ann Arbor, Mich.).

See also: Gerstenhaber, p. 870; Freudenthal, p. 871; Iséki, p. 872; Kostant, p. 930; Kastler, p. 973.

Lie Groups and Algebras

Raševskiĭ, P. K. A linear semisimple group as the group of invariance of a tensor of valency four. Trudy Sem. Vektor. Tenzor. Anal. **10** (1956), 105–117. (Russian)

Each semi-simple irreducible connected complex subgroup G_r (with Lie algebra G_r') of the general linear group GL(n, C) can be characterized by a tensor $a_{jq}{}^{ip}$ of valency four, which is preserved only by the matrices in G_r, and the scalar matrices. Let $L_\alpha=(L_{i\alpha}{}^j)$ $(\alpha=1, \cdots, r)$ be a basis for G_r'. Then $a_{jq}{}^{ip}=g^{\alpha\gamma}g^{\beta\delta}L_{j\alpha}{}^{i}L_{q\beta}{}^{p}L_{d\gamma}{}^{c}L_{c\delta}{}^{d}$, where $g_{\alpha\beta}$ is the Killing tensor of G_r'. For the result stated one shows that the space of tensors of the form $a_{jq}{}^{ip}b_p{}^q$ is identical with G_r'. The set of tensors $a_{jq}{}^{ip}$ arising in this fashion can be characterized by several symmetry conditions and a rank condition. *H. Samelson.*

Bruhat, François. Sur les représentations induites des groupes de Lie. Bull. Soc. Math. France **84** (1956), 97–205.

Let G be a separable locally compact group and let Γ_1 and Γ_2 be closed subgroups of G. For each unitary representation L_i of Γ_i let U^{L_i} denote the corresponding induced representation of G. When G is discrete (or more generally when Γ_1 and Γ_2 are open as well as closed) there is a theorem of the reviewer [Amer. J. Math. 73 (1951), 576–592; MR 13, 106] asserting that the intertwining number of U^{L_1} and U^{L_2} lies between two limits

which can be computed quite directly when L_1 and L_2 are known. These limits coincide in certain interesting cases (e.g. when L_1 and L_2 are one dimensional), and in these cases the theorem yields useful necessary and sufficient conditions for the irreducibility and unitary equivalence of U^{L_1} and U^{L_2}. In any event it yields sufficient conditions for irreducibility and non-equivalence. The reason for the rather strong assumption that Γ_1 and Γ_2 are open and closed is that the argument depends upon knowing that an arbitrary bounded linear operator from $\mathscr{L}^2(G/\Gamma_1)$ to $\mathscr{L}^2(G/\Gamma_2)$ is an integral operator and this is only true when the coset spaces G/Γ_1 and G/Γ_2 are discrete.

Now, when G is a (not necessarily connected) Lie group the operators in question can be realized as generalized integral operators whose kernels are distributions in the sense of Laurent Schwartz. Exploiting this fact the author is able to prove an analogue of the intertwining number theorem in which G is restricted to be a Lie group but in which the restrictions on Γ_1 and Γ_2 (and on L_1 and L_2) are greatly reduced. In particular he obtains a result strong enough to demonstrate the irreducibility of many of the most interesting representations of the semi-simple Lie groups.

The hypothesis that G is a Lie group not only makes possible a much better result about the irreducibility of induced representations than any previously known but also permits the development of a fairly extensive theory of induced representations in which L and U^L need not be unitary or even act in Banach spaces. The author's theorems on intertwining numbers and irreducibility are thus not presented as part of the existing theory of induced representations but are imbedded in a new theory in which one deals with more restricted groups and more general representations of them. This new theory is presented in considerable detail and its development occupies a large part of the paper. It has applications to the older theory in that there exist unitary representations which are induced by non-unitary ones.

The paper is divided into seven sections. The first is devoted to preliminaries of various sorts: differentiability of Radon-Nikodym derivatives, tensor products of topological vector spaces, distributions with a C_∞ manifold for domain and a topological vector space for range, convolutions of vector-valued distributions defined on Lie groups. The second section defines the class of representations to be considered and presents some of its more basic properties. When the space E in which the operators U_x of the representation $x \to U_x$ act is not "tonnelé" the usual strong continuity requirement is supplemented by the additional requirement that the U_x be equicontinuous as x varies in an arbitrary compact subset of G. When G is a Lie group the author defines for each representation U an associated "differentiable" representation U^0. The space of U^0 is the set E^0 of all ϕ in E for which $x \to U_x(\phi)$ is infinitely differentiable re-topologized so as to be complete. When $U=U^0$ the representation U is said to be differentiable. The transition from U to U^0 plays an important role in the author's later considerations. The third section is concerned with the existence and uniqueness of "quasi invariant distributions" on a variety M acted upon by a Lie group G. An E multiplier on M relative to G is a function A from $M \times G$ to operators on E such that $A(m, e)=1$ and $A(m, xy)=A(my^{-1}, x)A(m, y)$ for all m in M and all x and y in G. An E distribution T on M is said to be quasi invariant with multiplier A if $\int_M f(mx)dT(m)=\int_M A^{-1}(m, x)f(m)dT(m)$ whenever f is a C_∞ function with compact support and values in E.

Section four contains the author's definition of induced representation together with proofs of a few of the more elementary properties of the notion. First the induced representation U^L is defined as a differentiable representation of G whenever L is a differentiable representation of the closed subgroup g of G, the space of U^L being the set of all C_∞ functions f from G to the space of L satisfying the two conditions: (i) The image of the support of f in G/g is compact; (ii) $f(\xi x)=\rho(\xi)^{\frac{1}{2}} L_\xi(f(x))$ for all ξ, x in $g \times G$, where ρ is the canonical homomorphism of g into the positive reals defined by the Haar measures in g and G. More generally one passes to the differentiable representation L^0 associated with L and extends the resulting U^{L^0} by changing the topology in its space and then completing. It is to be noted that the U^L so defined is not uniquely determined by L but depends in addition upon which topology one introduces in the space of U^{L^0}. When L is unitary there is a canonical way of extending U^{L^0} to be unitary and one thus obtains U^L in the sense of the older theory. When L acts in a Banach space U^L can be taken so as to act in a Banach space—perhaps in many ways. In section five the author develops the analogues in his theory of the results on systems of imprimitivity and the representations of semi-direct products given previously for the older theory by the reviewer [Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 537–545; MR **11**, 158]. Due to the ambiguity in the definition of U^L the author's results are somewhat less sharp than those in the unitary case. A given representation of G is not shown to be equivalent to an induced representation but only to have a dense invariant subspace in which it is algebraically equivalent to the representation defined by a dense subspace of the representation induced by a differentiable representation. The representations concerned are assumed to yield "tempered representations" when restricted to the Abelian normal subgroup being studied. Here tempered means being not too badly unbounded in a precise sense suggested by Schwartz's definition of tempered distribution.

Section six contains the author's results on intertwining numbers and irreducibility criteria referred to at the beginning of this review. In contradistinction to those of section five most of the results of this section are new and interesting even in the unitary case. The principal result is the following. Let L be an irreducible unitary representation of a closed subgroup Γ of a Lie group G such that there are only countably many $\Gamma:\Gamma$ double cosets. If for each x in $G-\Gamma$ and each non-negative integer r there exists no irreducible component of the representation $\omega \to A_r(\omega)$ of $\Gamma_x=\Gamma \cap (x\Gamma x^{-1})$ which is a quotient of the representation $\omega \to L^0{}_\omega \otimes L^0{}_{x^{-1}\omega x}$ of Γ_x in $E^0 \otimes \bar{E}^0$, then U^L is irreducible. A_r is defined in terms of the distributions on the homogeneous space

$$(\Gamma \times (x\Gamma x^{-1}))/\tilde{\Gamma}_x,$$

where $\tilde{\Gamma}_x$ is the diagonal in $\Gamma_x \times \Gamma_x$, in a manner too complicated for reproduction here. The A_r did not occur in the previously known theorem in which Γ was assumed to be open. Their role was played by a single trivial representation — the one dimensional identity. Amongst the other results of this section are a necessary and sufficient condition for the irreducibility of U^L when Γ is a normal subgroup and a version of the Frobenius reciprocity theorem. This latter irreducibility criterion is an exact analogue of the one for finite groups given in the American Journal article cited above.

In the seventh and final section the results of section six are applied to discuss the irreducibility of the unitary representations of the semi-simple Lie groups which correspond to the members of the principal supplementary and degenerate series in the work of Gelfand and Neumark on the classical complex groups [Trudy Mat. Inst. Steklov. **36** (1950); MR **13**, 722]. It is shown in each case that almost all of the representations concerned are irreducible. The question of the irreducibility of the exceptional representations is left open with the conjecture that they are all in fact irreducible. It is remarked that Gelfand and Neumark and Harish-Chandra have shown that this is indeed the case for several important classes of groups.

In part the contents of this paper have been summarized in a series of notes [C.R. Acad. Sci. Paris **237** (1953), 1478–1480; **238** (1954), 38–40, 437–439, 550–553; **240** (1955), 2196–2198; MR **15**, 398, 504, 505; **16**, 996].

G. W. Mackey (Cambridge, Mass.).

See also: Freudenthal, p. 871; Yokota, p. 918; Haefliger, p. 920; Mautner, p. 929; Nomizu, p. 931; Couty, p. 931; Hano and Matsushima, p. 934; Igusa, p. 936.

Topological Vector Spaces

Halperin, Israel; and Luxemburg, W. A. J. The Riesz-Fischer completeness theorem for function spaces and vector lattices. Trans. Roy. Soc. Canada. Sect. III. (3) **50** (1956), 33–39.

For a given length-levelling function λ defined on the measurable functions on a σ-field, there is, in addition to the "normal" assumptions concerning λ, a large list of special properties which can be axiomatized for λ. Some of these properties are mutually equivalent, among others there are one-way implications, etc. An important axiom is, for example:

$$(\mathrm{L5}'''):\quad \lambda\Big(\sum_1^\infty u_n\Big)<\infty \text{ if } \sum_1^\infty \lambda(u_n)<\infty.$$

Or, again, one of its consequences $(\mathrm{L5}^{(\mathrm{vi})})$: if $\sum_\infty^1 \lambda(u_n)<\infty$, there is some u with $\lambda(u)<\infty$ such that $u_1+\cdots+u_n\leqq v_n$, where $\lambda(|v_n-u|)=0$, all n.

One constructs a linear space by means of a λ-type norm imposed on Banach-space-valued functions, as follows: Let F be a linear space of B-valued functions f such that $\|f\|$ is measurable. Let $L^\lambda(B)\equiv L^\lambda(B,F)$ be the set of f such that $\lambda(f)\equiv\lambda(\|f\|)<\infty$. Then $L^\lambda(B)$ is complete if $(\mathrm{L5}^{(\mathrm{vi})})$ holds.

Similar results obtain for vector lattices with semi-norms analogously restricted. *B. Gelbaum.*

Deprit, André. Endomorphismes de Riesz. Ann. Soc. Sci. Bruxelles. Sér. I. **70** (1956), 165–183.

A Riesz endomorphism is a continuous linear operator u on a locally convex vector space, satisfying: (i) u is an open mapping, (ii) the range of u^n is closed for each n, (iii) the null-space of each u^n is finite dimensional, (iv) for some m, u^m and u^{m+1} have the same null-space, (v) for some p, u^p and u^{p+1} have the same range. These endomorphisms provide an abstract approach to the Riesz theory of completely continuous operators. The results are substantially the same as those obtained independently by M. Altman [Studia Math. **15** (1956), 131–135; MR **17**, 1226]. *M. Jerison* (Lafayette, Ind.).

Correl, Ellen; and Henriksen, Melvin. On rings of bounded continuous functions with values in a division ring. Proc. Amer. Math. Soc. **7** (1956), 194–198.

Let X be a completely regular topological space, A a topological division ring. Let $C(X,A)$, $C^*(X,A)$, $C_p(X,A)$ denote, respectively, the ring of all continuous (bounded continuous, continuous with a compact $\overline{f(X)}$) A-valued functions on X. If, for a subring $C'\subset C(X,A)$ and every maximal (two-sided) ideal M in C', C'/M is isomorphic with A, then Stone's theorem is said to hold for C'. Main result: if A is totally disconnected, then Stone's theorem holds for $C_p(X,A)$; If A is locally compact totally disconnected nondiscrete, then it holds for $C^*(X,A)$. It is pointed out that a theorem by Goldhaber and Wolk [Duke Math. J. **21** (1954), 565–569; MR **15**, 968] concerning $C^*(X,A)$ is not correct unless X is normal; counter-examples are given. Another example shows, essentially, that, R denoting the reals, $\beta R\neq\bigcup\beta N$ where the sum is taken over all closed countable discrete $N\subset R$.

M. Katětov (Prague).

Hukuhara, Masuo; et Sibuya, Yasutaka. Théorie des endomorphismes complètement continus. J. Fac. Sci. Univ. Tokyo. Sect. I. **7** (1957), 391–405.

An expository paper on linear spaces given in Rome at the National Mathematics Institute in April 1956. The first parts are concerned with linear spaces in general. The last part concerns completely continuous endomorphisms, where an operator is completely continuous if the image of some neighborhood of 0 is compact. *M. E. Shanks.*

Jakubík, Ján. On convergence in linear spaces. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **6** (1956), 57–67. (Slovak. Russian summary)

A real linear space with a convergence is called simply a "linear" space if it satisfies: (1) $\alpha_n\to\alpha$ implies $\alpha_n x\to\alpha x$, (2) $x_n\to x$ implies $\alpha x_n\to\alpha x$, (3) $x_n\to x$, $y_n\to y$ implies $x_n+y_n\to x+y$, (4) if $x_n\to x$, $\{x_{k_n}\}$ is a subsequence, then $x_{k_n}\to x$, (5) if $x_n\to x$, $x_n\to y$, then $x=y$. In the system $\mathfrak{P}$ of all linear spaces with the same underlying abstract linear space, a partial order is defined: $P_1\leqq P_2$ if $x_n\to x$ in P_1 implies $x_n\to x$ in P_2. It is shown that, for any $P\in\mathfrak{P}$, there exists a maximal $P_0\in\mathfrak{P}$ with $P_0\geqq P$; for any $P_1\in\mathfrak{P}$, the system of all $P\in\mathfrak{P}$ with $P\leqq P_1$ is a complete modular lattice. A linear space may have different "pathological" properties; examples are given where e.g. $x\neq0$, α_n non$\to\alpha$, $\alpha_n x\to\alpha x$; or $x_n\to0$ whereas, with $z_{2k}=x_k$, $z_{2k-1}=x_k$, $\{z_k\}$ is not convergent. *M. Katětov.*

Raikov, D. A. Bundles of hyperplanes in linear spaces. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 760–762. (Russian)

An "L-space" is a vector space X with a distinguished family of hypersubspaces $\mathscr{E}$ such that if $E_1,\cdots,E_k\in\mathscr{E}$ then any hypersubspace containing their intersection also belongs to $\mathscr{E}$, and $\bigcap E(E\in\mathscr{E})=(0)$. A hyperplane is a set $x+E(E\in\mathscr{E})$. A family of hyperplanes is centered if each finite subfamily has a non-void intersection. A bundle S is a maximal centered-family. Now, $\mathscr{E}$ naturally distinguishes a class X' of linear functionals f on X. Each S defines a linear functional on X'. By means of such concepts, Šmulian's theorem on the double polar of a weakly compact convex set, and the reviewer's description of the locally convex topologies in X' associated with X (and thus also Mackey's earlier result) can be easily obtained. It is observed, of course, that each L-space gives rise to one of Mackey's linear systems $(X_{X'})$ and conversely.

R. Arens (Los Angeles, Calif.).

Sobolev, V. I. On functions of elements of a partially ordered ring. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. no. 1 (1956), 39–42. (Russian)

Proof of a theorem announced earlier [Dokl. Akad. Nauk SSSR (N.S.) 91 (1953), 23–26; MR 15, 297].

E. Hewitt (Seattle, Wash.).

Silverman, Robert J. Means on semigroups and the Hahn-Banach extension property. Trans. Amer. Math. Soc. 83 (1956), 222–237.

The following extension theorem is proved (Theorem 2). Let Y be an ordered linear space, X a subspace containing large elements (for $y \in Y$ there is $x \in X$ with $x \geqq y$), G a semi-group of monotone (if $x \geqq 0$ then $g(x) \geqq 0$ for all $g \in G$) linear operators on Y, each of which carries X into itself, and let G have an invariant mean (a positive linear functional on the bounded functions on G which is invariant under left and right translation). Then a G-invariant ($f(x)=f(g(x))$ for $x \in X$, $g \in G$) monotone linear transformation f of X into an order-complete vector lattice V has an extension which is a transformation of the same sort carrying all of Y into V, provided the set of non-negative elements in V is closed in the strongest (=largest) locally convex topology for V. A number of variants and applications of this theorem are given. The condition that G have an invariant mean is essential; the existence of an invariant mean is deduced (Theorem 7ff.) from rather weak extension properties for G-invariant functions. After treating a number of examples, the author considers the problem of finding conditions on order which imply continuity of a linear transformation F of a linear topological space Y into an ordered linear topological space V. Thus (via the proof of Theorem 11), if the inverse under F of the positive cone in V contains a non-void open set, and if $\{x: -a \leqq x \leqq a\}$ is bounded for each $a \in V$, then F is continuous. Results of a related nature are obtained for monotone transformations of ordered Banach spaces, and applications are made.

J. L. Kelley (Berkeley, Calif.).

Lions, J. L. Ouverts m-réguliers. Rev. Un. Mat. Argentina 17 (1955), 103–116 (1956).

Let Ω be an open set in E^n and let $D(\Omega)$ be the space of infinitely often differentiable functions on Ω with complex values and compact support in Ω. Designate by $W^m(\Omega)$ the space of functions $u \in L^2(\Omega)$ such that, with $p=(p_1, \cdots, p_n)$, $|p|=p_1+\cdots+p_n$, the derivatives

$$D^p u = \partial^{|p|}/\partial x_1^{p_1} \cdots \partial x_n^{p_n},$$

belong to $L^2(\Omega)$ for every p with $|p| \leqq m$ (where the differentiations are taken in the sense of the distributions on Ω). Designate by $E^m(\Omega)$ the space of the functions $u \in L^2(\Omega)$ such that $D^p u$ belongs to $L^2(\Omega)$ for every p with $|p|=m$ (without any assumption concerning the intermediate derivatives). The author calls Ω "m-regular" if the spaces $W^m(\Omega)$ and $E^m(\Omega)$ coincide. Moreover, he says that Ω satisfies the condition (Hi) if every parallel to the x_i-axis meets Ω in intervals of length greater than a fixed number $E_i>0$. The author proves the following theorem [which was stated by him without proof in Acta Math. 94 (1955), 13–153, in particular p. 71; MR 17, 745]: Ω is m-regular if the condition (Hi) is satisfied for every $i=1, \cdots, n$. *A. Rosenthal* (Lafayette, Ind.).

Sz.-Nagy, Béla. Contributions en Hongrie à la théorie spectrale des transformations linéaires. Czechoslovak Math. J. 6(81) (1956), 166–176. (Russian summary)

The article contains an outline of the investigations of Hungarian mathematicians in the spectral theory of linear operators. A list of works dealing with the results discussed by the author is given on pp. 166–167.

Author's summary.

Tillmann, Heinz Günther. Analytische Fortsetzung in der Fantappièschen Theorie der analytischen Funktionale. Arch. Math. 8 (1957), 43–45.

Depuis 1946 le reviewer a donné à la théorie „locale" des fonctionelles analytiques une nouvelle systématisation, qui a été complètement subordonnée à la théorie des espaces localement convexes, à la suite de travaux de Köthe [J. Reine Angew. Math. 191 (1953), 30–49; MR 15, 132] et d'autres auteurs. Le reviewer a montré d'ailleurs [Portugal. Math. 12 (1953), 1–47; MR 14, 656] que, même pour une théorie „globale", la topologie de l'espace $\mathfrak{S}_1$ de fonctions analytiques de Fantappiè n'était pas convéniente: l'unicité du prolongement analytique n'y subsistait pas, même pour les fonctionelles linéaires. Dans son „Habilitationschrift" [à paraître dans Abh. Math. Sem. Univ. Hamburg] l'auteur a défini dans $\mathfrak{S}_1$ une bonne topologie (d'espace métrisable), faisant disparaître toute anomalie et donnant encore, sur chaque partie non vide C de la sphère de Riemann, par passage au quotient, l'espace $\mathfrak{A}(C)$ des fonctions „locales" sur C, considéré par le reviewer, Köthe, Grothendieck, etc. Dans la présente note, l'auteur insiste sur le caractère pathologique de l'ancienne topologie, en montrant que, par rapport à cette topologie, toute fonctionnelle linéaire dans un ouvert admet des prolongements analytiques non linéaires.

J. Sebastião e Silva (Lisbonne).

Hongo, Eishi. On left rings of certain *-algebras. Bull. Kyushu Inst. Tech. (Math., Nat. Sci.) no. 2 (1956), 1–15.

The author continues work begun in same Bull. no. 1 (1955), 19–22 [MR 18, 588] on left *-algebras: A is called a left *-algebra if A admits a conjugate linear continuous anti-automorphism $x \to x_*$ of period two and if, at the same time, the following three conditions are satisfied for any elements x, y, z in A; (1) A is a *-algebra over complexes and a pre-Hilbert space with inner product (x, y) such that $(xy, z)=(y, x_* z)$, (2) the left multiplication $y \to xy$ is continuous for fixed x, and (3) the set of all elements of the form xy is dense in A.

Left and right rings of A are defined as usual and it is shown that they are commutants of one another. Dimension theory of such rings is discussed, leading to the result that the left ring of a left *-algebra has no purely infinite component if and only if there exists a positive invertible operator M commuting with L and satisfying $M^2=JM^2S$, with J, S being particular conjugations of the right and left rings respectively. *E. L. Griffin, Jr.*

Pitcher, T. S. Positivity in H-systems and sufficient statistics. Trans. Amer. Math. Soc. 85 (1957), 166–173.

Let G be a separable locally compact unimodular group, $H=H(G)$ its H-system [cf. T. S. Pitcher, same Trans, 77 (1954), 481–489; MR 16, 597] and W the weakly closed ring of operators generated by left convolution operators in H. For $x \in G$ let $l(x)$ denote the operation of left translation by x. Theorem 1.1. Let π be a non-zero central projection in W sending the set of nonnegative elements of H into itself. Let K be the set of $x \in G$ such that $l(x)\pi=\pi$. Then K is compact normal and for bounded f we have $\pi f(x)=\int_K f(xk)dk$ a.e. (dk normalized). This is applied to give a result in the theory of sufficient statistics: Theorem 2.1. Let M be a dominated

invariant set of probability measures on G and $K(M)$ the set of x such that $l(x)m=m$ holds identically for $m \in M$. Then $K(M)$ is compact normal and the Borel sets A not moved by any $k \in K(M)$ constitute a best sufficient subfield for M; the corresponding conditional expectation for bounded measurable f is just πf as above. *J. G. Wendel.*

Warner, Seth. Weakly topologized algebras. Proc. Amer. Math. Soc. **8** (1957), 314–316.

In a previous paper [Pacific J. Math. **5** (1955) 1025–1032; MR **17**, 876] the author considered the weak topology $\sigma(E, E')$ on a linear algebra E determined by a total subspace E' of the algebraic dual of E. It is now proved that if multiplication in E is continuous, the topology is necessarily locally m-convex. It is also proved that if E is a commutative, semi-simple, self-adjoint, complex Banach algebra, and E' is its Banach space dual, then multiplication in E under $\sigma(E, E')$ is continuous (if and) only if E is finite-dimensional. *J. H. Williamson.*

Lyusternik, L. A. Certain questions in non-linear functional analysis. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 6(72), 145–168. (Russian)

This is an expository article on the problems and methods of nonlinear functional analysis. Among the topics treated are: the differential of a mapping from one function space into another; power series, differential equations, and implicit function theorems in function spaces; the branch point theory for functional equations developed by A. Liapounoff and E. Schmidt; the Leray-Schauder topological degree for mappings in function spaces; and a number of aspects of the Morse theory. *J. Cronin* (New York, N.Y.).

See also: Cotlar, p. 893; Browder, p. 902; Thyssen, p. 904; Dobryšman and Dyubyuk, p. 904; Maurin, p. 905; Braconnier, p. 907; Soeder, p. 913; Ladyženskiĭ, p. 914; Berberian, p. 914; Coddington, p. 915.

Banach Spaces, Banach Algebras

Wermer, John. Subalgebras of the algebra of all complex-valued continuous functions on the circle. Amer. J. Math. **78** (1956), 225–242.

Let γ be a closed analytic curve bounding a region $\mathfrak{R}$ on a Riemann surface, and suppose $\mathfrak{R}+\gamma$ is compact. Let $\mathfrak{A}$ be the set of all functions on γ that have continuous extensions to functions analytic on $\mathfrak{R}$. Since γ is topologically a circle, $\mathfrak{A}$ may be regarded as a subalgebra of the algebra C of all continuous functions on the circle. Theorem. $\mathfrak{A}$ is a maximal closed (proper) subalgebra of C. The proof is the same as that given by the author [Proc. Amer. Math. Soc. **4** (1953), 866–869; MR **15**, 440] for the case where $\mathfrak{R}+\gamma$ is contained in the plane. The bulk of the present paper is devoted to a derivation of the form of the general linear functional on C that vanishes on $\mathfrak{A}$ — the main device in the proof of the theorem. It is also shown that $\mathfrak{A}$, as an algebra, determines $\mathfrak{R}$ up to conformal equivalence. *M. Jerison* (Lafayette, Ind.).

Wermer, J. Function rings on the circle. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 173–175.

Let φ, f belong to the normed ring C of all continuous complex-valued functions on the unit circle S_0, and satisfy: (a) φ and f are analytic in an annulus containing S_0, (b) $\varphi' \neq 0$ on S_0, (c) φ and f together separate the points of S_0. Let A be the ring of all polynomials in φ and f, including the constants. Theorem. $\bar{A} \neq C$ (if and) only if A is contained in a subring of the type of $\mathfrak{A}$ considered in the paper reviewed above. *M. Jerison.*

Helson, Henry; and Quigley, Frank. Existence of maximal ideals in algebras of continuous functions. Proc. Amer. Math. Soc. **8** (1957), 115–119.

Let S be a compact Hausdorff space and $C(S)$ the Banach algebra of all complex-valued continuous functions on S. Let A be a closed subalgebra of $C(S)$. Each point in S induces a maximal ideal in A, namely the ideal of all f in A vanishing at that point. Call a maximal ideal of A new if it is not induced by any point in S. When S is the unit disk $|z| \leq 1$ and A is the algebra of all f in $C(S)$ which are analytic in $|z|<1$, then A is a closed subalgebra of $C(S)$ having no new maximal ideals. When S is the circle or the unit interval it is not known whether there exist any closed subalgebras which fail to have new maximal ideals. The authors give the following two conditions on a closed subalgebra A of $C(S)$, S the circle or interval, under which new maximal ideals do exist: (1) A contains a function $e^{i\varphi}$, where φ is real-valued and constant on no subinterval of S; and (2) A has a subset which separates points such that each f in the subset maps S on a plane set of Lebesgue measure 0. As a corollary of (2) the authors obtain the following statement: Let Γ be an arc or a simple closed curve in the space C^n of n complex variables. Assume the arc or curve is differentiable. Then either every continuous function on Γ is uniformly approximable by polynomials in the coordinates, or there exists a point q not on Γ such that $|P(q)| \leq \max_{x \in \Gamma} |P(x)|$ for all polynomials P in the coordinates. {Note: It is reasonable to conjecture that for differentiable arcs the first alternative always holds.} *J. Wermer* (Providence, R.I.).

Helson, Henry; and Quigley, Frank. Maximal algebras of continuous functions. Proc. Amer. Math. Soc. **8** (1957), 111–114.

Let S be a compact Hausdorff space and $C(S)$ the Banach algebra of all complex-valued continuous functions on S. Let A be a closed subalgebra of $C(S)$ which is maximal, i.e. contained in no properly larger proper closed subalgebra of $C(S)$. The authors show the following. (1) If A does not contain the constant 1, then A is a maximal ideal. (2) If A does not separate points on S, then for some pair of points p and q with $p \neq q$ A is the set of all f in $C(S)$ with $f(p)=f(q)$. (3) Assume A contains 1 and separates points on S. Then there is a non-empty closed subset S_0 of S (which may equal S) and a maximal closed subalgebra B of $C(S_0)$ such that (i) A consists exactly of all functions f in $C(S)$ whose restriction to S_0 lies in B, and (ii) B has no divisors of zero. These results show that in studying maximal subalgebras one may assume, without real loss of generality, that the algebra is an integral domain with unit and that distinct points in S induce distinct points in the maximal ideal space of the algebra. An interesting problem raised by these results is whether, when S is a given compact space, $C(S)$ has any maximal subalgebras without divisors of zero. The answer is yes when S is the unit circle and is unknown when S is the full disk. The authors also show that a maximal subalgebra A without divisors of zero has the following

properties: (a) A contains no non-constant real function; (b) if a function f in A vanishes on an open set, then it vanishes identically; and (c) A is integrally closed, in a certain restricted sense, in its field of quotients.

J. Wermer (Providence, R.I.).

Vidav, Ivan. Eine metrische Kennzeichnung der selbstadjungierten Operatoren. Math. Z. **66** (1956), 121–128.

Let $\mathfrak{B}$ be a complex Banach algebra with identity 1, $\|1\|=1$. Let H' be the set of $u \in \mathfrak{B}$ for which

$$\|1+i\xi u\|=1+o(\xi)$$

as $\xi\to 0$, ξ real. Then H' is a real Banach space. If $u, v \in H'$, then $i(uv-vu) \in H'$, $u+iv=0$ only if $u=v=0$, and the spectrum of u is real. Moreover, if $u \in H'$ and $[\lambda, \varkappa]$ is the smallest interval of the real axis containing the spectrum of u, then for $\xi\geqq 0$, $\|e^{i\xi u}\|=1$, $\|e^{\xi u}\|=e^{\xi\varkappa}$, and $\|e^{-\xi u}\|=e^{-\xi\lambda}$.

Suppose now that $\mathfrak{B}$ contains a subset H satisfying: (A) Every element in $\mathfrak{B}$ can be written as $u+iv$ with $u, v \in H$; (B) $H\subseteq H'$; and (C) if $h \in H$ then there are $u, v \in H$ such that $h^2=u+iv$, $uv=vu$. Then $H=H'$, an involution can be defined on $\mathfrak{B}$ by $(u+iv)^*=u-iv$ for $u, v \in H$, and $\|u^2\|=\|u\|^2$ for $u \in H$. Moreover, if $\mathfrak{B}$ is given the equivalent norm $\|x\|_0=\|x^*x\|^{\frac{1}{2}}$, then B becomes a C*-algebra. The only use of (C) is to prove that if $h \in H$ then $h^2 \in H$. Along the way, the author proves a generalization of Kadison's generalized Schwarz inequality [Ann. of Math. (2) **56** (1952), 494–503; MR **14**, 481].

J. A. Schatz (Storrs, Conn.).

Bessaga, C. Bases in certain spaces of continuous functions. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 11–14.

Let $C(Q/H)$ denote the Banach space of all real functions defined and continuous on a compact metric space Q and equal to zero on a closed subset $H\subseteq Q$ and with the usual uniform norm. Let D be a cube in n-dimensional Euclidean space or a Hilbert cube, i.e., the set of all sequences $\{x_n\}$ in real separable Hilbert space such that $0\leqq x_n\leqq 1/n$ $(n=1, 2, \cdots)$. The author shows that $C(D/H)$ has a Schauder basis with norm 1. The norm of a basis $\{e_n\}$ of a Banach space is the least number K such that

$$\|t_1e_1+\cdots+t_pe_p\|\leqq K\|t_1e_1+\cdots+t_qe_q\|$$

for arbitrary scalars $\{t_k\}$ and arbitrary integers $p\leqq q$.

This result also follows from a more general theorem of F. S. Vakher [Dokl. Akad. Nauk SSSR (N.S.) **101** (1955), 589–592; MR **16**, 1031] which was obtained independently and at the same time as the above result. *A. Devinatz.*

Èskin, G. I. On a minimum problem in L-space. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 547–549. (Russian)

A theorem is given, with a condensed proof, which is an analogue for the space $L(a, b)$ of a theorem of R. Rado [Math. Z. **63** (1956), 486–495; MR **17**, 767] for the corresponding sequence space. The enunciation is: let $x_1(t)$, $x_2(t)$, $\cdots$, $x_m(t)$ be linearly independent elements of $L(a, b)$, and $c_1, c_2, \cdots, c_m$ complex numbers. Then, in the set Φ of all α satisfying $\int_a^b \alpha(t)x_i(t)dt=c_i$, there exists a unique element γ satisfying the following. 1) There is an integer n $(1\leqq n\leqq m+1)$ and a partition of (a, b) into nonintersecting sets $N_1, \cdots, N_n$ of positive measure such that $|\gamma(t)|=\rho_\nu$ almost everywhere on N_ν $(\nu=1, 2, \cdots, n)$, and $\rho_1>\rho_2>\cdots>\rho_n$. 2) For any α in the set Φ, there is an integer ν $(0\leqq\nu<n)$ such that $\alpha(t)=\gamma(t)$ almost everywhere in $N_1\cup N_2\cup\cdots\cup N_\nu$ and

$$\rho_\nu\geqq \text{ess. sup}\,\{|\alpha(t)|: t \in N_{\nu+1}\cup\cdots\cup N_n\}>\rho_{\nu+1}.$$

The function γ is in a sense a minimal solution of the equations. *J. L. B. Cooper* (Cardiff).

Sobolev, V. I. On the splitting of linear operators. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 951–954. (Russian)

Let E be a Banach space, E^* its dual, H a Hilbert space, such that $E\subset H\subset E^*$, $\|x\|_{E^*}\leqq a\|x\|_H$, for $x \in H$, $\|x\|_H\leqq b\|x\|_E$ for $x \in E$, for some constants a, b, and such that if $\{x, y\}$ denotes the value of $y \in E^*$ for $x \in E$, $\{x, y\}=(x, y)$ when $y \in H$ and (x, y) is the Hilbert scalar product. Let an operator A on E^* to E be selfadjoint, i.e. $\{Ax, y\}=\{x, Ay\}$ for x, y in E, and positive, i.e. $\{Ax, x\}\geqq 0$ for all x in E^*. Then it has an extension A' on E^{***} to E^{**}, and $A'=BB^*$ where B^* is on E^{***} to H, B on H to E^{**}. In particular, if E is a space L^p over a set of finite measure, an extension of a theorem of Krasnosel'skiĭ [same Dokl. (N.S.) **82** (1952), 333–336; MR **13**, 661] is obtained. *J. L. B. Cooper* (Cardiff).

Krasnosel'skiĭ, M. A.; and Rutickiĭ, Ya. B. General theory of Orlicz spaces. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. no. **1** (1956), 3–38. (Russian)

A general and thorough exposition of the theory of Orlicz spaces is presented. The main theorems of the subject are stated and proved and many interesting illustrations are provided. Although in its service as a synthesis the paper merits attention, its value to the experienced analyst in the field is limited since few new results (save for elegant reformulations) are given.

B. Gelbaum (Minneapolis, Minn.).

Šilov, G. E. On certain problems of the general theory of commutative normed rings. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 246–249. (Russian)

The author draws attention in this note to a number of important unsolved problems in the theory of commutative Banach algebras. First Problem: Let R be a commutative Banach algebra with unit e, and $\mathfrak{M}(R)=\{M\}$ the structure space of R (i.e., the space of all multiplicative linear functionals on R). For $x \in R$, let $\hat{x}$ be the function on $\mathfrak{M}$ such that $\hat{x}(M)=M(x)$. For any finite set $x_1, \cdots, x_n$ of elements in R, let $S(x_1, \cdots, x_n)$ be the set

$$\{(M(x_1), \cdots M(x_n)):M \in \mathfrak{M}\}$$

in n-dimensional complex space. Let f be a function defined and analytic on $S(x_1, \cdots, x_n)$. Does there exist an element $y \in R$ such that $M(y)=f(M(x_1), \cdots, M(x_n))$ for all $M \in \mathfrak{M}$? The author himself provided an affirmative answer to this query if $x_1, \cdots, x_n$ are generators of R [Mat. Sb. N.S. **32**(74) (1953), 353–364; MR **14**, 884, 1278]. Arens and Calderón [Ann. of Math. (2) **62** (1955), 204–216; MR **17**, 177] have published a proof of the affirmative answer for arbitrary $x_1, \cdots, x_n$. The author points out an error in their argument.

Second problem: The algebra R is called symmetric if for every $x \in R$ there is an element $\tilde{x} \in R$ such that $M(\tilde{x})=\overline{M(x)}$ for all $M \in \mathfrak{M}$. The algebra R is called antisymmetric if $x \in R$ and $\tilde{x} \in R$ imply that $x=\lambda e$ for some complex λ. Suppose that the norm of R is equivalent to the uniform norm of $\hat{x}$. Let R_1 be the subalgebra of R consisting of all elements x for which $\tilde{x}$ exists. For M_1, $M_2 \in \mathfrak{M}$ write $M_1\sim M_2$ if $M_1(x)=M_2(x)$ for all $x \in R_1$. This equivalence relation divides M into pairwise dis-

joint closed subsets, K. On each set K, the functions $\hat{x}$ form a Banach algebra R_K, and the author has shown [Ukrain. Mat. Žurnal 3 (1951), 404–411; MR **14**, 884] that R itself can be built up from the algebras R_K. That is, if f is continuous on $\mathfrak{M}$ and agrees with some x on each set K, then f has the form $\hat{y}$ for some $y \in R$. He raises the question of building up R in some similar way from antisymmetric algebras of functions defined on subsets of $\mathfrak{M}$. This problem is of considerable interest and is completely unsolved.

Third problem: Let G be a compact Abelian group and R a translation-invariant algebra of continuous complex-valued functions on G, with a norm such that

$$\|f(xa)-f(x)\|\to 0$$

as $a\to$ the group identity. For $b \in G$ and $f \in R$, let

$$\|f\|_b = \inf\{\|g\|: g=f \text{ in a neighborhood on } b\}.$$

Suppose that $\|f\|=\sup\{\|f\|_b: b \in G\}$. Then R is called a homogeneous algebra of functions of type C. The author has found all such algebras of functions on the circle group that contain all infinitely differentiable functions [Uspehi Mat. Nauk (N.S.) **6** (1951), no. 1(41), 91–137; MR **13**, 139]. Even for the 2-dimensional torus, the number of homogeneous algebras of type C becomes enormously larger. The author has found some such algebras [Dokl. Akad. Nauk SSSR (N.S.) **82** (1952), 681–684; MR **14**, 385], and he raises the question of finding out more about homogeneous algebras of type C on compact groups in general and on the torus in particular. Sample question: what are all non-affinely isomorphic homogeneous algebras of type C on the torus that contain all functions with continuous second partial derivatives?

E. Hewitt (Seattle, Wash.).

Gel'fand, I. M. On subrings of the ring of continuous functions. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 249–251. (Russian)

This note was prompted by and is an extension of the note reviewed above. Notation and terminology are as in the preceding review. The author first points out that the problem of finding all uniformly closed subalgebras of $C([0, 1])$, even those whose structure space is $[0, 1]$, is enormously complicated. He then poses a number of questions about antisymmetric algebras R for which the norm is equivalent to the uniform norm of $\hat{x}$. It follows from Šilov's theorem on the decomposition of R into the direct sum of ideals if $\mathfrak{M}(R)$ is disconnected [Mat. Sb. N.S. **32**(74) (1953), 353–364; MR **14**, 884, 1278] that no antisymmetric R can have zero-dimensional structure space. Question 1. Is there an antisymmetric algebra R with 1-dimensional structure space (e.g., a closed interval)?

For a bounded connected open set G in the complex plane, let $A(G)$ be the algebra of all functions analytic on G and continuous on $\bar{G}$. It is well-known that $\bar{G}$ is the structure space of $A(G)$; hence $A(G)$ is antisymmetric. Question 2. Is every antisymmetric algebra with 2-dimensional structure space isomorphic to some $A(G)$, under a homeomorphism of its structure space? Question 3. Is $A(G)$ a maximal antisymmetric algebra with structure space $\bar{G}$? Question 4. Do there exist antisymmetric algebras with structure space either the 2-sphere or the torus? No example is known of an antisymmetric algebra with 3-dimensional structure space. The analytic functions of two complex variables on a bounded connected open set in two-dimensional complex space, with continuous extensions over the closure of this set, are another example of an antisymmetric algebra. Questions like 2 and 3 are raised in connection with this example.

The author raises one final question: do the concepts of antisymmetry and analyticity coincide for algebras of continuous functions with uniform convergence?

E. Hewitt (Seattle, Wash.).

Soeder, Heinrich. Beiträge zur Funktionentheorie in Banachschen Räumen. Schr. Math. Inst. Univ. Münster **9** (1956), i+46 pp.

This thesis is concerned with some generalizations of theorems on analytic functions to functions f on X to Y, where X and Y are linear normed complete (B)-spaces relative to complex numbers. It is usually assumed that f is F-holomorphic [see E. Hille, Functional analysis and semi-groups, Amer. Math. Soc. Colloq. Publ., v. 31, New York, 1948, p. 72 ff; MR **9**, 594; also A. E. Taylor Math. Ann. **115** (1938), 446–484], i.e., that there exists a function $\delta f(x, h)$ on XX to Y, linear and continuous in h, such that $\lim_{\|h\|\to 0}[f(x+h)-f(x)-\delta f(x, h)]/\|h\|=0$, which implies G-holomorphism, i.e. $\lim_{\zeta\to 0}[f(x+\zeta h)-f(x)]/\zeta=\delta f(x, h)$. There is given an obvious extension of the implicit function and inverse function theorems contained for the real field in T. H. Hildebrandt and L. M. Graves, Trans. Amer. Math. Soc. **29** (1927), 127–153. The continuation theorem of Hartogs-Levi in this setting becomes: If X_1, X_2, Y are (B)-spaces, $y=f(x_1, x_2)$ is F-holomorphic for $\|x_1\|=r$, $\|x_2\|=0$, as well as for $\|x_1\|\leqq r$, $x_2=x_2^{(k)}$ with $\lim_k \|x_2^{(k)}\|=0$, then $f(x_1, x_2)$ is F-holomorphic for $\|x_1\|\leqq r$, $\|x_2\|=0$. There is also an extension of the Hartogs theorem mentioned by I. Shimoda and K. Iseki [J. Osaka Inst. Sci. Tech. Part I. **1** (1949), 61–66; MR **11**, 368] in the form: If $f(x_1, x_2)$ is F-holomorphic in x_1, x_2 on the boundary of the di-cylinder D: $\|x_1\|\leqq r$, $\|x_2\|\leqq r$, then f has an F-holomorphic extension into the interior of D. The last chapter concerns the case where X is the space of infinitely many variables: l^p: $1\leqq p\leqq\infty$. For $1\leqq p<\infty$, if f is F-holomorphic, then $\delta f(x, h)=\sum_{n=1}^{\infty} \partial f/\partial z_n\cdot h_n$, with absolute convergence on the right hand side. For $p=\infty$, this absolute convergence still holds, but the expression does not always give $\delta f(x, h)$ for all h.

T. H. Hildebrandt.

Lowdenslager, D. B. On postulates for general quantum mechanics. Proc. Amer. Math. Soc. **8** (1957), 88–91.

In Segal's "Postulates for general quantum mechanics" [Ann. of Math. (2) **48** (1947), 930–948; MR **9**, 241] a system of observables for quantum mechanics is characterized in terms of a squaring operation, a Banach space structure, and relations between them (where order is introduced by letting sums of squares be non-negative). In the course of a recent paper [ibid. **64** (1956), 593–601; MR **18**, 625] the reviewer has shown that the same ordered Banach space, namely $C(X)$, where the cardinal number of X is greater than two, can support different systems of observables, one with the conventional "commutative" square and another with a non-conventional "non-commutative" square. The paper under review, using essentially the same technique, shows how any non-trivial system of observables, "commutative" or not, can be given a "non-commutative" squaring operation retaining the original ordered Banach structure. In order to carry out this program the author formulates Segal's system in terms of the ordered Banach space structure with no explicit reference to squaring. Let a partially ordered Banach space with unit be a partially ordered linear space with an element $e>0$, which is a Banach space under the norm $\|x\|=\min\{\lambda: -\lambda e\leqq x\leqq \lambda e\}$, where

the minimum is attained. A spectral function L on a partially ordered Banach space S with order unit e is a function assigning to each point $x \in S$ a closed subspace $L(x)$ of S such that: 1) $e \in L(x)$, 2) $L(x)$ is a lattice in the ordering it inherits as a subspace of the partially ordered Banach space S, 3) $v \in L(x)$ if and only if $L(v)$ is a sublattice of $L(x)$, and 4) the function sending $x \in S$ into $x \vee 0 \in L(x)$ is continuous in the norm on S. Theorem: Under the situation described above there is a unique squaring operation $x \to x^2$ satisfying Segal's postulates, which generates the same positive cone, the same unit e, and such that each $L(x)$ is a commutative subsystem. Corollary: In any partially ordered Banach space with order unit e there is at least one spectral function L satisfying the theorem and hence a square satisfying Segal's postulates. An $L(x)$ which satisfies the theorem and which was used by the reviewer in the case of $C(X)$ is: $L(e)$=the linear space generated by e, and, for $x \neq e$, $L(x)$=the 2-dimensional space generated by x and e.

S. Sherman (Philadelphia, Pa.).

Ladyženskiĭ, L. A. On a class of non-linear equations. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. no. 2 (1956), 31–38. (Russian)

Motivated by the need to study operators A of the form: $A\varphi(x)=\int K(x, y, \varphi(y))dy$, the author continues his investigations in the field of operators over partially ordered Banach spaces. Included are two types of theorems, those on the non-existence and those on the existence of solutions of equations of the form $A\varphi+f=\lambda\varphi$, for f in the positive cone K which defines the partial ordering in the space E. For example: (1) Let $A\varphi \geqq Q\varphi$, $\varphi \in K$, where Q is a linear, completely continuous u_0-bounded ($\exists u_0 \in K$ $(u_0 \neq \theta)$ such that $\theta \neq \varphi \in K \Rightarrow \exists$ an integer p and α, $\beta>0$ such that $\alpha u_0 \leqq Q^p\varphi \leqq \beta u_0$) operator. If λ_Q is a positive eigenvalue of Q, then $A\varphi+f=\lambda\varphi$, $f \geqq \theta$, has no solution in K if $\lambda<\lambda_Q$. (2) In the same notation, if A is asymptotic to Q $(\lim_{\|\varphi\|\to\infty}\|A\varphi-Q\varphi\|/\|\varphi\|=0)$ then there are positive solutions of the equation if $\lambda>\lambda_Q$.

There are also sections on u_0-concave operators and on specializations in which the partial ordering is continuous.

B. Gelbaum (Minneapolis, Minn.).

Berberian, S. K. The regular ring of a finite AW^*-algebra. Ann. of Math. (2) **65** (1957), 224–240.

There is associated with any complemented modular lattice L (of order $\geqq 4$) a regular ring R whose lattice of principal ideals is isomorphic to L [von Neumann, Continuous geometry, part II, Inst. Advanced Study, Princeton, 1937]. This result applies to the projection lattice of any AW^*-algebra A [Kaplansky, Ann. of Math. (2) **53** (1951), 235–249; MR **13**, 48]. In general the connection between A and R involves only the projection lattice of A and so ignores much of the algebraic structure of A. On the other hand, for the special case in which A is a finite ring of operators, R is concretely given as a set of closed densely defined operators which contains A as a self-adjoint subalgebra [Murray and von Neumann, ibid. **37** (1936), 116–229; von Neumann, loc. cit. supra]. In the present paper, the author extends this result to the case of finite AW^*-algebras. He gives a new construction for R which applies to the abstract case. The abstract analogue of the unbounded closed operators are pairs of sequences (x_n, e_n) of elements of A, where the e_n are projection (self-adjoint idempotents) such that $e_n \leqq e_{n+1}$, sup $e_n=1$, $x_ne_m=x_me_m$, and $x_n{}^*e_m=x_m{}^*e_m$, for $m<n$. Two such objects (x_n, e_n) and (y_n, f_n) are defined to be "equivalent" if there exists another sequence $\{g_n\}$ of projections such that $x_ng_n=y_ng_n$ and $x_n{}^*g_n=y_n{}^*g_n$, for all n. The equivalence class containing (x_n, e_n) is denoted by $[x_n, e_n]$. These are the elements of R. The algebraic operations in R are defined as follows:

$$[x_n, e_n]+[y_n, f_n]=[x_n+y_n, e_n \cap f_n],$$
$$\lambda[x_n, e_n]=[\lambda_n{}^*, e_n],\ [x_n, e_n]^*=[x_n{}^*, e_n]$$

and $[x_n, e_n][y_n, f_n]=[x_ny_n, g_n]$ for an appropriately chosen sequence $\{g_n\}$. The embedding of A in R is given by $x \to [x, 1]$. The image of A in R consists of the "bounded" elements of R; i.e. $\|x_n\| \leqq M$, for all n. The ring R is characterized by the property that it is a regular complex $*$-algebra with unity 1 which contains A as a self-adjoint subalgebra such that $x^*x+y^*y+z^*z=1$ implies $x, y, z \in A$.

C. E. Rickart (New Haven, Conn.).

See also: Singer, p. 891; Massera, p. 900; Silverman, p. 910; Sz.-Nagy, p. 910; Warner, p. 911; Feldman and Fell, p. 915; Aronszajn and Panitchpakdi, p. 917.

Hilbert Space

Sunouchi, Haruo. A characterization of the maximal ideal in a factor. II. Kōdai Math. Sem. Rep. **7** (1955), 65–66.

This corrects a mistake in the first paper in this series [same Rep. **1954**, 7; MR **15**, 968]. The result is as follows: Theorem 1. Any factor M of infinite type, except for the countably decomposable case, has an unique maximal ideal. This ideal consists of operators A in M such that each spectral projection of $\sqrt{(A^*A)}$ has relative dimension strictly smaller than that of the identity operator.

E. L. Griffin, Jr. (Ann Arbor, Mich.).

Daleckiĭ, Yu. L.; and Kreĭn, S. G. Integration and differentiation of functions of Hermitian operators and applications to the theory of perturbations. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. no. **1** (1956), 81–105. (Russian)

The proof and applications are given of a formula for the differential coefficient of a function $f(H(t))$, where $H(t)$ is a bounded Hermitian operator defined and differentiable for t in some segment of the real line, and f a function with continuous second derivatives on a segment (a, b) of the real line containing the spectrum of $H(t)$ for all t in the segment of definition. The formula is

$$\frac{df(H(t))}{dt}=\int_a^b\int_a^b \frac{f(\lambda)-(f\mu)}{\lambda-\mu}\, dE_\lambda(t)\, \frac{dH(t)}{dt}\, dE_\mu(t),$$

where $E_\lambda(t)$ is the spectral projector of $H(t)$.

The formula is extended to higher order derivatives by iteration, and a Taylor's theorem, with the integral form of the remainder, is proved. The differential coefficients of the operator functions and the integrals are understood to exist in the sense of convergence by operator norms; the properties of the Stieltjes integral with respect to the spectral function are discussed in some detail. By taking f to be a function equal to 1 on an isolated portion of the spectrum and zero elsewhere on the spectrum of H, applications are given to perturbation theory: the formulae simplify considerably when $H(t)$ is linear in t. Estimates for the error term in the Taylor's expansion are given. The principal results have appeared without proof in Dokl. Akad Nauk SSSR (N.S.) **76** (1951), 13–16; Dopovidi Akad. Nauk Ukrain. RSR **1951**, 234–238 [MR **12**, 617; **16**, 264].

J. L. B. Cooper (Cardiff).

Gonshor, Harry. Spectral theory for a class of non-normal operators. Canad. J. Math. 8 (1956), 449–461.

An operator A on a Hilbert space H is a J_n operator if there exists a direct integral decomposition of H such that A is decomposed into copies of an operator of order less than or equal to a cardinal number n (pure J_n refers to equality). J_1, J_2 classes are equivalent to normal, binormal, respectively [A. Brown, Amer. J. Math. 76 (1954), 414–434; MR 15, 967]. Theorem 1: A is a pure J_n operator if and only if H can be decomposed into n orthogonal equivalent projections E_i with corresponding matrix units V_{ij} (partial isometries) such that $A=\sum_{ij} A_{ij}U_{ij}$, where the A_{ij} are commuting normal operators and commute with $U_{k,l}$ for k, l different from i, j.

The main result of the paper is a "spectral theorem" for J_2 operators: Let Z be the complex numbers ordered lexicographically with respect to polar coordinates r, θ ($z=re^{i\theta}\in Z$, $0\leq\theta<2\pi$) and let Q be the set of triples (λ, μ, a), where $\lambda, \mu\in Z$, $\lambda\geqq\mu$, a real and positive. Now, if A is a J_2, then there exists a direct integral decomposition onto $Z\cup Q$ such that A is decomposable and such that $A(t)$ equals either copies of λ at $\lambda\in Z$ or copies of $\begin{pmatrix}\lambda & a\\ 0 & \mu\end{pmatrix}$ at (λ, μ, a). A multiplicity theorem is also proved.

E. L. Griffin, Jr. (Ann Arbor, Mich.).

Coddington, Earl A. On self-adjoint ordinary differential operators. Math. Scand. 4 (1956), 9–21.

Let $L=p_0D^n+p_1D^{n-1}+\cdots+p_n$ $(D=d/dx)$ be a formally self-adjoint ordinary differential operator with complex-valued coefficients p_k having $(n-k)$ continuous derivatives on an open interval (a, b) and $p_0\neq0$ on (a, b). Let D_S be the totality of C^{n-1} functions $u(x)$ with compact supports on (a, b) such that $u^{(n-1)}(x)$ is absolutely continuous and $Lu\in L_2(a, b)$. Under the assumption that S has a self-adjoint extension H, it is shown how to define a self-adjoint boundary-value problem on a finite closed interval δ of (a, b) in such a way that, as $\delta\to(a, b)$, every convergent sequence of the corresponding Green functions $G_\delta(x, y, \lambda)$ tends to the same limit $G(x, y, \lambda)$ uniformly on any compact (x, y, λ)-region with $\Im(\lambda)=0$. The resolvent $G(\lambda)=(H-\lambda I)^{-1}$ of H is defined by the integral operator with this kernel $G(x, y, \lambda)$. In this way, the general expansion theorem of Weyl-Stone-Titchmarsh-Kodaira type for H is obtained as the limiting case of those corresponding to δ.

K. Yosida (Tokyo).

Coddington, Earl A. On maximal symmetric ordinary differential operators. Math. Scand 4 (1956), 22–28.

This is an extension of the paper reviewed above. Every maximal symmetric extension H of S has a unique generalized resolvent: $G(\lambda)=(H^*-\lambda I)^{-1}$ for $\Im(\lambda)>0$ and $=(H-\lambda I)^{-1}$ for $\Im(\lambda)<0$. It is defined by an integral operator with the kernel $G(x, y, \lambda)$ which is the limit, as $\delta\to(a, b)$, of every convergent subsequence of the Green functions $G_\delta(x, y, \lambda)$ corresponding to the compact subintervals δ of (a, b) and with a suitable self-adjoint boundary condition. $G(\lambda)$ is associated with a generalized resolution of the identity, in the sense of Neumark [cf. N. L. Achieser and I. M. Glasman, The theory of linear operators in Hilbert space, Gostehizdat, Moscow-Leningrad, 1950; MR 13, 358; 16, 596], so that a general expansion theorem for H, similar to the case of self-adjoint H, is obtained as in the paper reviewed above.

K. Yosida (Tokyo).

Coddington, Earl A. Generalized resolutions of the identity for closed symmetric ordinary differential operators. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 638–642.

This is another extension of the paper reviewed second above. Let T_0 be the smallest closed symmetric operator in $L_2(a, b)$ associated with the given formally self-adjoint ordinary differential operator L. Let $\{F(\lambda)\}$ be any generalized resolution of the identity, in the sense of Neumark, associated with T_0. The corresponding generalized resolvent $R(\lambda)$, $\Im(\lambda)\neq0$, is defined by

$$(R(\lambda)f, g)=\int_{-\infty}^{\infty}(\mu-\lambda)^{-1}d(F(\mu)f, g).$$

It is shown that $R(\lambda)$ is also defined by an integral operator of Carleman type. Writing this kernel as the sum of a certain fundamental solution of $(L-\lambda I)u=0$ and a second kernel, $F(\lambda)$ is obtained explicitly by inverting the generalized resolvent. In this way a general expansion theorem is obtained for T_0.

K. Yosida (Tokyo).

Solomon, Liviu. Sur un théorème de Friedrichs relatif à l'extension des opérateurs positifs définis. Com. Acad. R. P. Romîne 6 (1956), 627–634. (Romanian. Russian and French summaries)

Feldman, J.; and Fell, J. M. G. Separable representations of rings of operators. Ann. of Math. (2) 65 (1957), 241–249.

Let A be a ring of operators (weakly closed self-adjoint algebra of operators containing the identity) on a Hilbert space H. A "representation" of A is any algebraic $*$-homomorphism of A into the algebra of all bounded operators on some Hilbert space K, which carries the identity of A into the identity operator. The representation is called "separable" if both H and K are separable. A directed set $\{A_\nu\}$ of bounded operators on H is said to be "σ-weakly convergent to A" if

$$\lim_\nu \sum (A_\nu x_i, y_i)=\sum (Ax_i, y_i)$$

for all $\sum \|x_i\|^2<\infty$ and $\sum \|y_i\|^2<\infty$. The main purpose of this paper is to show that, whenever A contains no finite direct summand or whenever A is a factor of Type II [Kaplansky, Ann. of Math. (2) 53 (1951), 235–249; MR 13, 48], then every separable representation of A is σ-weakly continuous. An important result is that a representation is σ-weakly continuous if, and only if, it is completely additive on projections. This follows from the fact that a positive linear functional on A is σ-weakly continuous if, and only if, it is completely additive on projections [J. Dixmier, Bull. Soc. Math. France 81 (1953), 9–39; MR 15, 539]. This observation makes it possible to carry out most of the discussion for the more general AW^*-algebras [Kaplansky, loc. cit.]. The condition in the abstract case which replaces separability of H is the "countable chain condition", which asserts that any pairwise orthogonal family of projections (self-adjoint idempotents) is countable. The principal results obtained are the following: (1) Let A, B be two AW^*-algebras and assume that B satisfies the countable chain condition. If A is either purely infinite with center satisfying the countable chain condition or is a finite factor of Type II, then every $*$-homomorphism of A into B is completely additive on projections. (2) If A is a finite Type II ring of operators on a separable space H and B is a finite ring of operators on a separable space K, then any $*$-homomorphism of A into B is completely additive on projec-

tions. The authors conjecture that any separable representation of a finite Type II ring of operators is completely additive on projections. (3) Not every closed 2-sided ideal in a finite Type II AW^*-algebra A is an intersection of maximal 2-sided ideals. However, if B is any AW^*-algebra satisfying the countable chain condition, then the kernel of any $*$-homomorphism of A into B is an intersection of maximal ideals. *C. E. Rickart.*

Maslov, V. P. Theory of perturbations of linear operator equations and the problem of the small parameter in differential equations. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 531–534. (Russian)

The author states without proof several theorems about sequences $T_n f_n$, the f_n belonging to a separable Hilbert space, the T_n (unbounded) operators such that both T_n and T_n^* converge (strongly or uniformly). From these abstract functional-analytic results he deduces, also without proof, theorems about the behavior of solutions to non-homogeneous linear differential equations under certain small perturbations of the coefficients and right-hand side. For example, Theorem 2: Consider for each $\varepsilon>0$ the ordinary equation

$$\varepsilon y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_0y=f_\varepsilon,$$

with zero boundary conditions

$$y(x_0)=y'(x_0)=\cdots=y^{(n-1)}(x_0)=0.$$

Let a_m be m-times differentiable, and independent of ε; let $a_{n-1}\equiv a_{n-2}\equiv\cdots\equiv a_{k+1}\equiv 0$, but $a_k\neq 0$, with $n-k$ odd. Let the right-hand sides f_ε converge in $L^2(x_0, x_1)$ and let the set of squares of solutions y_ε be bounded in L^2. Then the y_ε converge in L^2 to the solution of the formal limit equation with zero boundary conditions. Theorem 5. Let the elliptic equation $\varepsilon\Delta u+c^2(x_1,\cdots,x_p)u=F_\varepsilon(x_1,\cdots,x_p)$ have the solution u_ε vanishing on the (smooth) boundary Γ of the domain Ω of c, F_ε, u_ε. Define $u_0=F_0/c^2$, suppose all functions in sight belong to $L^2(\Omega)$, and suppose u_0 bounded on Γ. Then u_ε converges to u_0 in the L^2 norm. These theorems extend and simplify some of the work of A. N. Tihonov and V. M. Volosov. *H. Mirkil.*

See also: Campanato, p. 905; Sz.-Nagy, p. 910; Hongo, p. 910; Pitcher, p. 910; Bochner, p. 940; Kastler, p. 973.

TOPOLOGY

General Topology

Hammer, Preston C. General topology, symmetry, and convexity. Trans. Wisconsin Acad. Sci. Arts Lett. **44** (1956), 221–255.

The author makes a careful study of a generalized notion of closure. Beginning with an arbitrary function f whose domain and range are families of subsets of a set M, a subset X of M is called f-closed if and only if $f(Y)\subset X$ whenever Y is a subset of X which belongs to the domain of f. There are corresponding generalizations of the notions of closure, limit, interior, derived set, etc. This very general approach enables the author to treat simultaneously closure, some types of symmetry, convexity and midpoint convexity, and several other concepts. *J. L. Kelley* (Berkeley, Calif.).

Kapuano, Isaac. Ensembles jouissant de la propriété de Baire au sens restreint. C. R. Acad. Sci. Paris **244** (1957), 708–711.

M désignant un ensemble linéaire jouissant de la propriété de Baire au sens restreint; propositions concernant les ensembles $\mathscr{G}\subset M$ qui sont des G_δ relativement à M, ainsi que les ensembles $\mathscr{C}\subset M$ qui jouissent de la propriété C. Décompositions non dénombrables de M. (Résumé de l'auteur). *L. Gillman* (Lafayette, Ind.).

Smiley, M. F. Filters and equivalent nets. Amer. Math. Monthly **64** (1957), 336–338.

The author introduces an equivalence relation between the nets in a given set in such a manner that there is a natural one-to-one correspondence between classes of equivalent nets and filters in the set (or between classes of equivalent filter-bases). An ordering between nets α, β is given such that if $\alpha\geqq\beta$, then β is equivalent to a subnet of α. *R. G. Bartle* (Urbana, Ill.).

Kuratowski, K. Sur le rôle des espaces abstraits en topologie moderne. Schr. Forschungsinst. Math. **1** (1957), 27–32.

An expository article. Both the "inner-mathematical" and the "practical" usefulness of "l'espace fonctionnel" Y^X are clearly illustrated.

Bagley, R. W. A note on topologies on 2^R. Michigan Math. J. **3** (1955–56), 105–108.

Let R be a set and let 2^R be the class of functions on R to the two element set $\{0, 1\}$. Topologies T_1, T_2, T_3 and T_4 for 2^R are defined by agreeing that a subset A is T_i-closed if and only if it is closed under: (T_1) sequential order convergence, (T_2) transfinite sequential order convergence, (T_3) net order-convergence, (T_4) the topology of pointwise convergence. Letting T_i be the class of complements of T_i-closed sets, it is known that $T_3=T_4$ (in fact, net order-convergence is precisely convergence relative to T_4) and the author observes that

$$T_1\supset T_2\supset T_3\supset T_4$$

and $T_1=T_4$ if R is countable. He then shows that 2^R is not T_2-compact if R has cardinal greater than or equal to that of the continuum, and hence, in this case, $T_2\neq T_3$. If R is uncountable then the class of finite subsets of R is T_2-dense in 2^R and $T_1\neq T_2$. Certain of these results are extended to the space of all maps of R into a complete lattice L.

There are incorrect statements in the paper (as noted in an errata sheet accompanying the index to the volume); but I believe that the major results, as stated above, are correct. *J. L. Kelley* (Berkeley, Calif.).

Powderly, Mary; and Tong, Hing. On orbital topologies. Quart. J. Math. Oxford Ser. (2) **7** (1956), 1–2.

For any transformation γ of a set S into S, a T_1-topology is defined which renders γ continuous and is, in general, non-trivial. Reviewer's remark: This topology may be described as follows. An open subbase consists of all sets $(x)\cup\{y|\gamma^m(y)=x$ for some $m=1, 2, \cdots\}$, as well as $(x)\cup\{y|\gamma^n(y)\neq x\}$, $n=1, 2, \cdots$. *M. Katětov* (Prague).

Henriksen, Melvin. On minimal completely regular spaces associated with a given ring of continuous functions. Michigan Math. J. **4** (1957), 61–64.

In this review, all functions are real-valued and continuous. For a completely regular space X, let μX be a dense subspace of X that is minimal with respect to the

property that every function on μX has an extension to X. In answer to a question of L. J. Heider, the author shows that, in general, no such μX exists, e.g., if X is the Stone-Čech compactification of an infinite discrete space. But if a μX exists, then it is unique: it consists of all isolated points of X together with all points p such that some function on $X\sim\{p\}$ has no extension to X. [See also, Daly and Heider, Bull. Amer. Math. Soc. **63** (1957), 37.] {On p. 62, l. 3, the first μX should read vX.} *M. Jerison.*

Davison, Walter F. An equivalence relation for compact Hausdorff varieties. Proc. Amer. Math. Soc. **7** (1956), 1109–1114.

By consistent use of the uniformities of general topology a remarkable extension is given of the concept of Fréchet equivalence, so much used in the modern calculus of variations for curves and surfaces and in differential geometry. A concept of equivalence ($\sim$) is established in any family $\mathfrak{F}$ of mappings f from a compact Hausdorff space X into any set Y. This set Y need not have a topology provided a compatibility condition is satisfied. If $\mathfrak{F}$ denotes the family of all closed subsets of X the compatibility condition is (*) $f^{-1}g(E)\in\mathfrak{F}$ for all $f, g\in\mathfrak{F}$ and all sets $E\in\mathfrak{F}$. For each $f\in\mathfrak{F}$ a topology $\mathfrak{F}_f$ can be defined in $f(X)\subset Y$, as usual, by saying that a set $E\in\mathfrak{F}_f$ provided $f^{-1}(E)\in\mathfrak{F}$. Then $f(X)$ with the topology $\mathfrak{F}_f$ is a compact Hausdorff space. Then the uniformity $\mathfrak{U}$ associated to X with the topology $\mathfrak{F}$ is considered, and, analogously, the uniformity $\mathfrak{U}_f$ associated to $f(X)$ with the topology $\mathfrak{F}_f$. The preliminary nontrivial statements follow: (i) for every $U\in\mathfrak{U}$ and $y\in f(X)$ the set $fUf^{-1}(y)$ is a $\mathfrak{F}_f$-neighborhood of y; (ii) the collection of relations in $f(X)\times f(X)$ defined by $W_f(U)=fUf^{-1}fUf^{-1}$, $U\in\mathfrak{U}$, is a base for the uniformity $\mathfrak{U}_f$. Let H be the family of all homeomorphisms of X onto itself. The relation ($\sim$) can now be defined as follows for any two mappings $f, g\in\mathfrak{F}$. It is said that $f\sim g$ provided for every $U\in\mathfrak{U}$ there exists a homeomorphism $h\in H$ such that $fh\subset W_g(U)g$. If $f\sim g$ then $f(X)=g(X)$, and $\sim$ is an equivalence relation.

If Y is a topological space with topology $\mathfrak{S}$ for which every compact set is closed, and $\mathfrak{F}$ is a class of continuous mappings, then it is proved that $\mathfrak{F}$ necessarily satisfies the compatibility condition (*), and the topology $\mathfrak{F}_f$ on $f(X)$ agrees with the relative topology $\mathfrak{S}_r$ obtained from $\mathfrak{S}$ relatively to $f(X)$. If Y is a separated uniform space with uniformity $\mathfrak{B}$, then it is proved that $f, g\in\mathfrak{F}$, $f\sim g$ if and only if for every $V\in\mathfrak{B}$ there exists a homeomorphism $h\in H$ such that $fh\subset Vg$. If X is a Peano space and Y a metric space with metric $d(y', y'')$, then it is proved that $f\sim g$ if and only if f, g are Fréchet equivalent. As usual, f, g are said to be Fréchet equivalent provided for every $\varepsilon>0$ there exists a homeomorphism $h\in H$ such that $d[fh(x), g(x)]<\varepsilon$ for all $x\in X$. *L. Cesari.*

Smirnov, Yu. M. On strongly paracompact spaces. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 253–274. (Russian)

The following four conditions are shown to be equivalent for a regular topological space R: every open covering of R has a star-finite (star-countable) open (closed) refinement (with "star-finite open", we have the well-known definition of strongly paracompact or S-spaces). Properties of metric S-spaces are considered [for some of them cf. K. Morita, Math. Japon. **1** (1948), 60–68; MR **10**, 204]. It is shown that, for a metrizable R, if there exists a closed continuous transformation f of R into B_τ such that every $f^{-1}(y)$ is separable, then R is an S-space; if R is an S-space, then there exists a continuous transformation of R into B_τ such that every $f^{-1}(y)$ is separable (B_τ denotes, essentially, the topological product of countably many discrete spaces of power τ). Examples are given showing that the above implications cannot be reversed. *M. Katětov* (Prague).

Aronszajn, N.; and Panitchpakdi, P. Extension of uniformly continuous transformations and hyperconvex metric spaces. Pacific J. Math. **6** (1956), 405–439.

The following extension problem is investigated: Given a uniformly continuous map T of a metric space D into a metric space F, when is there a uniformly continuous extension $T^\sim$ to every possible metric space E which contains D metrically? It is shown that such extension for all spaces E implies that T has a sub-additive modulus of continuity. Restricting attention to maps having such a modulus, a necessary and sufficient condition on F that an extension $T^\sim$ be always possible, with $T^\sim$ having the same modulus of continuity as T, is that F be hyperconvex, in the sense that: for every class $\{S(x_i, r_i), i\in I\}$ of closed spheres in F (the ith sphere of radius r_i about the point x_i) such that the distances $\rho(x_i, x_j)\leq r_i+r_j$ for i and j in I, it is true that $\bigcap S(x_i, r_i)$ for $i\in I$ is non-void. (For totally convex spaces this can be rephrased: each family of closed spheres, each two of which intersect, has a non-void intersection). For each cardinal m, a notion of m-hyperconvexity is defined by restricting the families of spheres to have cardinal less than m. A careful study is then made of m-hyperconvexity, its relationship to m-separability, and to completeness. A hyperconvex space F has the (topological) property of being a retract of each containing metrizable space in which it is closed. An investigation of hyperconvex Banach spaces is then made, and, following some interesting propositions on convex sets, it is shown that the unit sphere of such a space has an extreme point. This furnishes the affirmative answer to a problem of Nachbin, and in conjunction with a theorem of Nachbin [Trans. Amer. Math. Soc. **68** (1950), 28–46; MR **11**, 369], shows that every hyperconvex Banach space is isometric to a space $C(H)$ of all continuous real-valued functions on an extremally disconnected Hausdorff space H. [This result was proved directly by the reviewer by different methods, **72** (1952), 323–326; MR **13**, 659.] Finally, the authors characterize neatly those spaces H for which $C(H)$ is m-hyperconvex. *J. L. Kelley* (Berkeley, Calif.).

Alexandroff, Paul. Die Kontinua ($\mathbf{V}^p$)—eine Verschärfung der Cantorschen Mannigfaltigkeiten. Monatsh. Math. **61** (1957), 67–76.

Let X denote a compact Hausdorff space whose covering dimension is n. The statement that X is a Cantorian manifold can be defined as follows: if A and B are disjoint closed subsets of X with non-empty interiors and C is a closed subset of $X-(A\cup B)$ separating A from B in X, there is a finite open covering ω of X such that no mapping of C into a space of dimension less than $n-1$ is subordinate to ω; i.e., has the property that each point inverse is contained in some element of ω. The author strengthens this notion in defining a class $\mathbf{V}^n$ of spaces as follows: X belongs to $\mathbf{V}^n$ if and only if, for each two disjoint closed subsets A and B of X with non-empty interiors, there is a finite open covering ω of X such that, if C is a closed subset of $X-(A\cup B)$ separating A from B in X, then no mapping of C into a space of dimension less than

$n-1$ is subordinate to ω. He observes that every compact subspace of $(n+1)$-dimensional Euclidean space R^{n+1} which is the common boundary of two domains in R^{n+1}, and all classical compact n-manifolds, belong to $\mathbf{V}^n$. He then proves that every compact metric n-dimensional space contains a subspace which belongs to $\mathbf{V}^n$. Several unsettled questions are raised. *E. Dyer.*

Proskuryakov, I. V. Construction of the spectrum of a compact space containing a given topological space of the same dimension. Mat. Sb. N.S. **39**(81) (1956), 219–238. (Russian)

The note contains new proofs of some well-known results of dimension theory. A construction is given, for any separable metrizable n-dimensional R, of a sequential spectrum [in the sense of P. Alexandroff, Ann. of Math. (2) **30** (1928), 101–187] defining an n-dimensional compactum containing R. The question is raised of the possibility of modifying this construction, as well as P. Alexandroff's definition, so as to obtain R itself as a limit (in an appropriate sense) of the spectrum.

M. Katětov (Prague).

Fort, M. K., Jr. Extensions of mappings into n-cubes. Proc. Amer. Math. Soc. **7** (1956), 539–542.

The author generalizes some theorems concerning mappings f with $\dim f^{-1}(y)\leq k$ [for $k=0$, cf., e.g., Katětov, Časopis Pěst. Mat. Fys. **75** (1950), 1–16; Czechoslovak. Math. J. **2**(77) (1952), 333–368; MR **12**, 119; **15**, 815]. Let X denote a separable metric space, K a closed subset of X, h a mapping of K into the n-cube I_n, $C_n(h|K)$ the set of all mappings f of X into I_n with $f(x)=h(x)$ on K. It is proved that if $A\subset X-K$ is relatively closed, $\dim A=m\geq n$, then the set of all $f\in C_n(h|K)$ of type $m-n$ on A (i.e. such that $\dim(A\cap f^{-1}(y))\leq m-n$ for every $y\in I_n$) is a residual set in $C_n(h|K)$. From this lemma, it is easily obtained that e.g. if h is of type $m-n$ (on K), then there exists $f\in C_n(h|K)$ of type $m-n$ (on X); if X is compact, $m\geq n$, then $\dim X\leq m$ if and only if there exists a mapping f of X into I_n which is of type $m-n$. *M. Katětov.*

Nagami, Keiô. Some theorems in dimension theory for non-separable spaces. J. Math. Soc. Japan **9** (1957), 80–92.

If R is a topological space, let $\dim R$ be the covering dimension of R, $\operatorname{ind} R$ be the usual (small) inductive dimension of R, and $\operatorname{Ind} R$ be the (large) inductive dimension of R based on boundaries of neighborhoods of closed sets. The author proves, among other items: (1) The sum theorem for metric spaces. (2) If R is paracompact Hausdorff, S hereditarily paracompact Hausdorff, φ a closed map of R on S, then $\dim R\leq\sup_{y\in S}\dim\varphi^{-1}(y)+\operatorname{Ind} S$. (3) If $\dim S=0$ and S is paracompact Hausdorff, and R, φ are as in (2), then $\dim R\leq\sup_{y\in S}\dim\varphi^{-1}(y)$. Finally, the author defines a covering-dimension kernel and a large-inductive-dimension kernel both of which are better adapted to non-separable spaces. Several theorems involving the Čech compactification of these dimension kernels are proved. *M. E. Shanks.*

See also: Schmidt, p. 868; Gerstenhaber, p. 870; Cesari, p. 882; Zahorska, p. 885; Timan, p. 890; Correl and Henriksen, p. 909; James, p. 918; Freudenthal, p. 921.

Algebraic Topology

★**Pontrjagin, L. S. Grundzüge der kombinatorischen Topologie.** Hochschulbücher für Mathematik, Band 29. VEB Deutscher Verlag der Wissenschaften, Berlin, 1956. vii+133 pp.

A translation of the book reviewed in MR **11**, 450, an English translation of which is listed in MR **14**, 196.

Yokota, Ichiro. On the cellular decompositions of unitary groups. J. Inst. Polytech. Osaka City Univ. Ser. A. **7** (1956), 39–49.

The author describes a cellular decomposition (in a generalized sense) of the $SU(n)$ from which one can read off immediately the Betti numbers of $SU(n)$, and which leads quite simply to the cohomology ring structure, and the Steenrod squares and powers in $SU(n)$ and in the complex Stiefel manifolds. The basic cell e^{2n-1} in $SU(n)$ is obtained by mapping the suspension $E(M_n)$ of complex projective n-space into $SU(n)$, utilizing the map which to the complex unit vector Y and the complex number a of form $\cos\theta\cdot e^{-i\theta}$ associates the unitary transformation which sends the vector X into $X-2a(X, Y)Y$. The cells $e^3, e^5, \cdots, e^{2n-1}$ and their Pontryagin products (in this order) form all the cells for $SU(n)$; the e^k are primitive cycles. Analogous cells were used for $SO(n)$ by J. H. C. Whitehead [Proc. London Math. Soc. (2) **48** (1944), 243–291; MR **6**, 279] and C. E. Miller [Ann. of Math. (2) **57** (1953), 90–114; MR **14**, 673]; for the Steenrod powers by A. Borel and J. P. Serre [Amer. J. Math. **75** (1953), 409–448; MR **15**, 338]. The cells e^k appear also in E. B. Dynkin, Dokl. Akad. Nauk SSSR (N.S.) **91** (1953), 201–204 [MR **15**, 398]. *H. Samelson* (Ann Arbor, Mich.).

Zeeman, E. C. A proof of the comparison theorem for spectral sequences. Proc. Cambridge Philos. Soc. **53** (1957), 57–62.

In this paper the author proves a comparison theorem for spectral sequences which generalizes earlier known results. We suppose given a spectral sequence $\{E^r\}$, $r\geq 2$, satisfying the following conditions: 1) E^r is bigraded, and $E_{p,q}^r=0$ if p or q is less than zero; 2) $d^r:E_{p,q}^r\to E_{p-r,q+r-1}^r$; 3) there is an exact sequence

$$0\to E_{p,0}^2\otimes E_{0,q}^2\to E_{p,q}^2\to \operatorname{Tor}(E_{p-1,0}^2, E_{0,q}^2)\to 0;$$

4) A is a graded filtered group such that $E^0(A)=E^\infty$. Let $B_p=E_{p,0}^2$, $B=\sum B_p$, $C_q=E_{0,q}^2$, and $C=\sum C_q$. Theorem: If $\{E^r\}$ and $\{\bar{E}^r\}$ are spectral sequences satisfying the preceding conditions, and $f^r:E^r\to\bar{E}^r, r\geq 2$, $f^A:A\to\bar{A}$, $f^B:B\to\bar{B}$, and $f^C:C\to\bar{C}$ are maps compatible with the preceding structure, then if any two of f^A, f^B, and f^C are isomorphisms so is the third.

Actually more general theorems are proved which provide that if any two of f^A, f^B, and f^C are isomorphisms up to some given dimension, then the third is an isomorphism up to a dimension depending only on the dimension involved in the first two isomorphisms.

J. C. Moore (Princeton, N. J.).

James, I. M. Commutative products on spheres. Proc. Cambridge Philos. Soc. **53** (1957), 63–68.

Soit la sphère S^n, $n\geq 1$; on considère l'ensemble $I(n)$ des entiers q pour lesquels existe une application continue $f: S^n\times S^n\to S^n$ telle que $f(x, y)=f(y, x)$ et que l'application $x\to f(x, e)$ de S^n dans S^n soit de degré q. On démontre: $I(n)=\{0\}$ si n est pair; si n est impair, $I(n)$ se compose de tous les multiples d'une certaine puissance de 2, soit

$2^{r(n)}$. On a

$$r(1)=0,\ r(3)=2,\ 3\leq r(n)\leq n-1 \text{ pour } n \text{ impair } \geq 5,$$
$$r(n)-1\leq r(n+m)\leq r(n)+m \text{ pour } n \text{ impair, } m \text{ pair } \geq 2.$$

Les démonstrations reposent sur les propriétés homologiques et homotopiques du „carré symétrique" L^n de S^n; en particulier, on utilise le fait que $\pi_{n+3}(L^n)=Z_3$ pour n impair ≥ 3 (résultat dû à Nakaoka pour $n\geq 5$).

H. Cartan (Paris).

Kosiński, A. On mappings which satisfy certain conditions on boundary. Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 335–340.

The general problem is: given a map f of (A, A_0) into $(B, \mathrm{Fr}B)$, where B is a closed subset of a compact space M, deduce properties of f_*: $H_k(A)\to H_k(B)$ from properties of f_{0*}: $H_k(A_0)\to H_k(\mathrm{Fr}B)$ and from homological properties of M. (Homology is Čech, with a compact coefficient group.)

Among the results are: (1) If M is acyclic in dimensions k and $k+1$, and if f_{0*} is onto or has a right inverse, then so does f_*. A dualized form of this theorem, for subsets of manifolds, is also given. (2) Let A and B be closed subsets of S_n, $A_0=\mathrm{Fr}A$, $f(\mathrm{Int}\,A)=\mathrm{Int}\,B$, and let f be damping on Int A in the sense that if y is near FrA then the diameter of $f^{-1}[f(y)]\cap\mathrm{Int}\,A$ is small. (It is shown that this is always the case if f_0 maps FrA homeomorphically onto FrB).) Then, if f_{0*} is an isomorphism onto for each k, it is true that f induces an isomorphism of the homology sequence of $(A, \mathrm{Fr}A)$ onto that of $(B, \mathrm{Fr}B)$.

The proof of (2) leans heavily on results of Sitnikov [Mat. Sb. N.S. **31**(73) (1952), 439–458; MR **14**, 489].

J. L. Kelley (Berkeley, Calif.).

★**Spanier, E. H.; and Whitehead, J. H. C. The theory of carriers and S-theory. Algebraic geometry and topology.** A symposium in honor of S. Lefschetz, pp. 330–360. Princeton University Press, Princeton, N.J., 1957. $7.50.

This paper contains the details of the so called S-theory developed by the authors and announced earlier [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 655–660; MR **15**, 52]. The main idea of S-theory is to study the so called suspension category instead of the category of topological spaces and continuous maps. In its simplest form, the objects of the suspension category are topological spaces, but the maps are not continuous maps or homotopy classes of continuous maps. If X and Y are topological spaces a map from X to Y in the suspension category is represented by a continuous map f: $S^nX\to S^nY$ for some n, where S^nX is the n-fold suspension of X, and S^nY is the n-fold suspension of Y. If f: $S^nX\to S^nY$ and g: $S^mX\to S^mY$, then f is equivalent to g if there exists an integer k such that $S^{k-n}f$: $S^kX\to S^kY$ and $S^{k-m}g$: $S^kX\to S^kY$ are homotopic. The maps in the suspension category are the equivalence classes there defined. In the suspension category the maps from one object to another always form an abelian group. For this and other reasons, the suspension category is in general a such simpler category to deal with than the category of topological spaces and continuous maps.

The foregoing description of S-theory is incomplete and naive. The main reason for this is that all questions involving carriers have been ignored. Such questions are treated in detail by the authors, and enable them to define many more invariants than would be possible in the theory outlined. Analogues of homotopy group, relative homotopy group, triad homotopy group etc. are among them.

Although this paper has only recently appeared, part at least of its contents have been current for a considerable period and many applications of the theory by the authors and others have been made. The applications to duality theory are among the most important [Spanier and Whitehead, Mathematika **2** (1955), 56–80; MR **17**, 653].

J. Moore (Princeton, N.J.).

Peterson, Franklin P. Some results on cohomotopy groups. Amer. J. Math. **78** (1956), 243–258.

In this paper the author generalizes many of the previous known results on cohomotopy groups. The main result is that the Hopf classification theorem for mappings of a finite complex into a sphere holds in C-theory [J. P. Serre, Ann. of Math. (2) **58** (1953), 258–294; MR **15**, 548], i.e. modulo certain classes of groups. More precisely, let C be a class of abelian groups such that if $A, B\in C$, then $A\otimes B\in C$, and $\mathrm{Tor}(A, B)\in C$. Further, let (K, L) be a pair of CW-complexes each of which is of dimension less than or equal to N. Denote by $H^q(K, L)$ the q-dimensional cohomology group of the pair (K, L), and $\pi^q(K, L)$ the q-dimensional cohomotopy group when it is defined. Under these conditions the following theorem is valid. Suppose that $n>\frac{1}{2}(N+1)$ and $H^q(K, L)\in C$ for $q>n$. Then there is an exact sequence for each $q>\frac{1}{2}(N+1)$,

$$0\to A^q\to\pi^q(K, L)\xrightarrow{\eta^q}H^q(K, L)\to B^q\to 0,$$

such that A^q and B^q belong to C if $q>\max\{\frac{1}{2}(N+1), n-\alpha\}$, where α is a positive integer depending on the class C. An immediate corollary of this theorem is that the cohomotopy groups of a finite complex are finitely generated

J. C. Moore (Princeton, N.J.).

Oniščik, A. L. Spaces of paths and fiber spaces. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 932–935. (Russian)

To every fiber space E, in the sense of Serre, with base B, fiber F, projection p, the author associates a new fiber space, whose total space is F, whose base is E, and whose fiber is $\Omega(B)$, the space of loops in B, based at some point b_0. Actually the total space is not F itself, but a space homotopically equivalent to it, namely the fiber space induced from the fiber space of all paths in B starting at b_0, (the Serre-space of B), by the projection p of E into B. Similarly the fiber is not actually $\Omega(B)$, but the equivalent space of all paths in E, starting at a point x_0 and ending in a fixed fiber F. {Reviewer's note: another construction for this associated space consists in taking for total space the space of all paths in E, starting in F, and projecting into E by sending the path into its endpoint.} The construction can be iterated, giving rise to $(\Omega(B), F, \Omega(E))$, then to $(\Omega(E), \Omega(B), \Omega(F))$, etc.; certain assumptions about connectedness have to be made. The spectral sequences of the first three associated spaces and of the Serre-spaces of B, E, F are connected by a sequence of homomorphisms; the maps are the obvious ones, in case the E_2-terms are tensor products. The construction is used to prove several theorems about fiber spaces, e.g.: If $H(E)$, $H(B)$ and $H(F)$ are exterior algebras, then $H(E)=H(B)\otimes H(F)$ (the proof uses the fact that in the third associated fiber space all degrees are even). For a homogeneous space G/U of a Lie group there is equivalence between: $U\not\sim 0$ in G; $H^*(M)$ is an exterior algebra; $H^*(\Omega(M))$ is a polynomial algebra [rational coefficients].

H. Samelson (Ann Arbor, Mich.).

Haefliger, André. Sur l'extension du groupe structural d'une espace fibré. C. R. Acad. Sci. Paris **243** (1956), 558–560.

To a homomorphism φ of the topological group H onto G corresponds an operation which to each H-principal bundle E' over the base B associates a G-principal bundle E over B, namely "division" by the kernel N of φ. The inverse question, given φ and E, to find E', is the extension problem. Following A. Borel and J. -P. Serre [Amer. J. Math. **75** (1953), 409–448; MR **15**, 338], this leads to the question of a cross-section of a bundle over B, induced from a "universal" bundle with the classifying spaces B_H, B_N, B_G as bundle, fiber, base. If H is connected and N discrete, existence of the section depends on a characteristic class in $H^2(B, N)$, the transgression of the characteristic element in $H^1(G, N)$ of the covering of G by H. E.g. for extending from SO(n) to Spin(n), the second Stiefel-Whitney class must vanish. If H is compact connected Lie, one is led to a characteristic element in $H^2(B, \bar{N})$, where $\bar{N}$ is a finite subgroup of the center of N, determined by the extension H of G by N [cf. A. Shapiro, Ann. of Math. (2) **50** (1949), 581–586; MR **11**, 157].

H. Samelson (Ann Arbor, Mich.).

Noguchi, Hiroshi. On regular neighbourhoods of 2-manifolds in 4-Euclidean space. I. Osaka Math. J. **8** (1956), 225–242.

The author finds (Theorem 3) that an orientation preserving homeomorphism between orientable 2-manifolds in Euclidean 4-space can be extended to an orientation preserving homeomorphism between regular neighborhoods of them if and only if it can be done locally. He then attempts to show that any collection of local singularities can occur, but his proof is incorrect and the result (Lemma 5.8 and Theorem 4) is in contradiction with unpublished results of J. W. Milnor and the reviewer. In a letter to the reviewer the author mentioned some other errors in the paper but said that these were remediable.

R. H. Fox (Princeton, N.J.).

Dowker, C. H. Imbedding of metric complexes. Algebraic geometry and topology. A symposium in honor of S. Lefschetz, pp. 239–242. Princeton University Press, Princeton, N. J., 1957. $7.50.

The author proves: (1) Every metric complex is an absolute F_σ. (2) A metric complex is an absolute G_δ if and only if the following condition C is satisfied: each sequence $a_1, a_2, \cdots$, where a_i is a proper face of a_{i+1}, is finite. As a consequence of (2), he obtains (3): A metric complex is an ANR for collectionwise normal spaces if and only if the condition C is satisfied.

J. Dugundji.

See also: Cartier, p. 870; Fox, p. 885; Burdina, p. 933; Bernard, p. 933.

GEOMETRY

Geometries, Euclidean and other

Niče, Vilko. Die Brennachsenkongruenz der Zylinder eines Kreises. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II. **11** (1956), 37–44. (Serbo-Croatian summary)

Considered are the ∞^2 quadratic cylinders passing through a circle c (center O). Through O, vertical to the plane of c, passes the axis of a circular cylinder. Each of the other ∞^2 cylinders have elliptic normal cross sections, which determine two real and two imaginary focal axes. The subject of the paper is the congruence of the ∞^2 real focal axes, which arrange themselves on ∞^1 hyperboloids of rotation. This congruence is found to be of order two and of class two. Several other properties of this congruence are derived.

S. R. Struik.

Amir-Moéz, A. R. Synthetic approach to the theory of the envelope. Amer. Math. Monthly **64** (1957), 265-268.

Hofmann, J. E. Der sog. Lehrsatz des Ptolemaios als Flächensatz. Math. Naturwiss. Unterricht **9** (1956/57), 452–454.

Adad, H. Sur la résolution d'un triangle, connaissant les longueurs de trois bissectrices issues respectivement des trois sommets. Publ. Sci. Univ. Alger. Sér. A. **2** (1955), 159–171 (1957).

The author determines the three sides a, b, c of a triangle when three of the bisectrices are given originating resp. in the three vertices; when d_a, d_b, d_c are the inner, d_a', d_b', d_c' the outer bisectrices, a graphic method facilitates the solution of $(b+c)^2 \cdot d_a^2 = bc(b+c+a)(b+c-a)$ (and mutatis mutandis other analogous equations) for a, b, c. The cases a) the three bisectrices are concurrent (or not) b) none of the three is infinite (at least one is infinite) are investigated separately. He introduces the concept "triangle of Study" defined by H. Wolff [J. Reine Angew. Math. **177** (1937), 134–151]. In the cases of the d_i concurrent and leading to an equation of 10th degree he cites the results of B. L. v. d. Waerden [ibid. **179** (1938), 65–68].

S. R. Struik.

Thébault, V. Systèmes de cercles et de points cosphériques. Mathesis **65** (1956), 418–421.

Four circles drawn in the four faces of a tetrahedron T are cospherical, if and only if each vertex of T has equal powers for the three circles located in the three faces passing through the vertex considered.

This proposition is due to M. Monseau [Mathesis **65** (1956), 281–282], and Thébault offers another proof for it. A second proposition, related to the above, also given by Monseau [loc. cit.] is due to Thébault [ibid. **42** (1928), 31–33, p. 32], who reproduces here his own proof. He also points out some applications of both properties considered.

N. A. Court (Norman, Okla.).

Thébault, V. Triangle bordé de carrés. Mathesis **65** (1956), 423–426.

Given a triangle $T=ABC$, let T' be the triangle formed by the centers of the three squares constructed on the side of T externally, and let T'' be the analogous triangle for the squares constructed on the sides of T internally.

The author proves several properties of the triangles T', T'', of which we may quote the following: The nine point center of the triangle T' (T'') coincides with the orthogonal center of the three circles inscribed in the three corresponding squares.

This property remains valid when the three points A, B, C are collinear, as was pointed out by J. Langr [Mathesis **65** (1956), 476–477].

N. A. Court (Norman, Okla.).

Naumann, Herbert. Über Vektorsterne und Parallelprojektionen regulärer Polytope. Math. Z. 67 (1957), 75–82.

Der Satz von Pohlke, wonach jedes zentrierte konvexe Sechseck die Parallelprojektion eines Würfels ist, wird in folgende Aussage verallgemeinert: Jedes Zonoeder (d.i. ein konvexes Polytop mit zentrierten zweidimensionalen Begrenzungsflächen) im R_n, das p verschiedene Kantenrichtungen aufweist, ist in einem R_d als Parallelprojektion des Würfels γ_p darstellbar, wobei $d=\max(p, 2n-1)$ ist. Die Herleitung verwendet die Begriffe des eutaktischen Vektorsterns [Coxeter, Regular polytopes, Methuen, London, 1948; MR 10, 261] und den Satz, dass jeder Vektorstern $\mathfrak{S}_p$ aus p Vektoren des R_n im R_d als Parallelprojektion eines dilatierten Einheitssternes des R_p darstellbar ist. Mit denselben Methoden beweist der Verf. u.a. folgende Sätze: Jedes konvexe Polytop des R_n mit p Ecken ist im R_d als Parallelprojektion eines regulären Simplexes α_{p-1} darstellbar, wobei $d=\max(p-1, 2n-1)$. Ferner ist jedes konvexe Mittelpunktpolytop des R_n mit $2p$ Ecken im R_d als Parallelprojektion des Oktaeders β_n zu erhalten, $d=\max(p, 2n-1)$. *J. J. Burckhardt.*

Hughes, D. R. Partial difference sets. Amer. J. Math. 78 (1956), 650–674.

The author defines partially transitive planes as follows. Suppose π is a finite projective plane of order n (i.e. every line of π has $n+1$ points), and G is a non-trivial group of collineations of π; let π_0 be the set of points and lines of π that are fixed (element-wise) by every collineation of G. Let the points and lines of π_0 be called fixed points and fixed lines; let the points (lines) of π that are not in π_0 but are on lines (contain points) of π_0 be called tangent points (tangent lines); let the remaining points (lines) be called ordinary points (ordinary lines). If G is transitive and regular on both the ordinary points and the ordinary lines, we call π a partially transitive and regular plane with respect to G and π_0.

For π_0 we have the following possibilities. (0) π_0 is empty. (1a) π_0 consists of a point Q_0 and a line K_0, Q_0 on K_0. (1b) π_0 consists of a point Q_0 and a line K_0, Q_0 not on K_0. (2) π_0 consists of two points Q_0 and Q_1 and two lines K_0 and K_1, where $K_0=Q_0Q_1$ and Q_0 is on K_1. (3) π_0 consists of three non-collinear points Q_i ($i=0, 1, 2$) and the 3 lines $K_0=Q_1Q_2$, $K_1=Q_0Q_2$, $K_2=Q_0Q_1$. (4, m) π_0 consists of m points ($m\geqq 3$) Q_i ($i=1, 2, 3, \cdots, m$) on a line K_0, a point Q_0 not on K_0, and the $m+1$ lines K_0, $K_i=Q_0Q_i$. (5, m) π_0 consists of $m+1$ points ($m\geqq 2$) Q_i ($i=0, 1, 2, \cdots, m$) on a line K_0 and $m+1$ lines K_i ($i=0, 1, \cdots, m$) each through Q_0. (6, m) π_0 is a sub-plane (non-degenerate) of order m, with points Q_i, lines K_i ($i=0, 1, \cdots, m^2+m$). In Section 5 the author finds an example of type (4, m) with $m=4$; in Section 6 he finds infinitely many examples of type (5, m); he has no examples of type (6, m) but has no proof that they cannot occur. There are both Desarguesian and non-Desarguesian examples of types (1a), (2) and (3). From numerous interesting results of the author we quote: Theorem 17. In type (4, m), if $m\neq 3$, then $n=(m-1)^2$. Theorem 21. In type (5, m), $n=m^2$. Theorem 23. In type (6, m), $n=m^4$.

Some of the author's methods seem hopeful for the construction of planes of non-prime-power order. *S. Chowla* (Boulder, Colo.).

Freudenthal, Hans. Kompakte projektive Ebenen. Illinois J. Math. 1 (1957), 9–13.

The v. Staudt's foundation of projective geometry was finished by adding the axioms of order and continuity. Afterwards non-trivial theorems of incidence were added in order to reduce the dependence on the order and the continuity. The axioms of order are evidently not appropriate for complex geometry. A middle way is to base the non-trivial axioms of incidence in the plane upon Desargues' theorem and then to add an axiom of continuity in the form of compactness and connectivity. We are led, by Desargues' theorem, to a geometry on a field (commutative or non-commutative). The author considers a topological projective plane, which is a continuum, as a topological space. He shows that the straight line and the plane are necessarily connected and contractible in the small, and that every proper closed subset of the straight line is contractible into a point on it, a family of homeomorphic mappings being admitted as the contractions. Also it is shown that the straight lines are spheres of the dimensions 1, 2, 4 and 8.

A set of axioms is given and then the theorem below is proved stepwise. Axioms: A topological projective plane E is a topological space, in which certain proper closed subsets are distinguished as straight lines so that the following holds. (1.1) For every two distinct points a, b, there exists just one straight line ab, which contains a and b. (1.2) For every distinct pair of lines there exists just one point which is contained in both. (1.3) When for every four points a, b, c and d, no three of which belong to one and the same straight line, the common point $\sigma(a, b, c, d)$ of the straight lines $\overline{ab}$ and $\overline{cd}$ is defined, then σ is a continuous function of the four arguments a, b, c and d. (Thus joining and intersecting are continuous operations.) (1.4) E is compact. (1.5) E has a positive dimension. Notation: The straight lines of E are homeomorphic to one another. Denote one of them by P.

Theorem: Every proper closed subset of P contracts into a point. [Thus, if p and q be two distinct points of P, then there exists a family of homeomorphisms f_t ($0\leqq t<1$) of P onto itself, such that $f_t(x)$ is a continuous function of t and x, that $f_t(p)=p$, $f_t(q)=q$, $f_t(P\setminus(p, q))\subset P\setminus(p, q)$, and that $\lim_{t\to 1} f_t(x)=p$ uniformly in every closed subset of P, which does not contain q.] *T. Takasu* (Yokohama).

Green, H. G. On the theorems of Ceva and Menelaus. Amer. Math. Monthly 64 (1957), 354–357.

Černyaev, M. P. On an invariant of a projective transformation. Rostov. Gos. Ped. Inst. Uč. Zap. no. 3 (1955), 133–134. (Russian)

The following theorem is proved: "If six planes belonging to one and the same bundle are intersected by an arbitrary seventh plane not belonging to the given bundle, then a certain ratio of products formed from the areas of triangles so obtained does not depend on the position of the intersecting plane." Compare the theorem of Menelaus in the plane.

Hohenberg, Fritz. Projektionen projektiver Räume. Monatsh. Math. 61 (1957), 54–66.

Verfasser zeigt, daß man durch Projektion eines reellen oder komplexen projektiven Raumes P_n aus einem Zentrum Z auf einen anderen Raum Π und eventueller Kombination mehrerer solcher Projektionen alle bisher in der Darstellenden Geometrie aufgetretenen Abbildungen erhält. Z und Π sind dabei irgendwelche Teilräume des P_n von den Dimensionen n_z und n_π. Bei der Untersuchung sind folgende Fälle zu unterscheiden: a) $n_z+n_\pi=n-1$, b) $n_z+n_\pi>n-1$, c) $n_z+n_\pi<n-1$. Zunächst wird der

Fall a) behandelt. Bei $n_\pi=2$ wird von einem Riß und bei $n_\pi=1$ von einer Kotierung gesprochen. Werden mehrere derartige Projektionen je auf die Räume $\Pi_1, \cdots, \Pi_k$ miteinander verknüpft, wobei $\sum_i \dim \Pi_i=N$ gesetzt wird, so ist wiederum zwischen den Fällen α) $N>n$, β) $N=n$, γ) $N<n$ zu unterscheiden. Bei Fall γ) ist der Raumpunkt umgekehrt nicht eindeutig aus seinem Bildpunkt zu konstruieren, bei β) ist das der Fall, während bei Fall α) Bedingungen zu erfüllen sind, damit zusammengehörige Punkte ein Urbild besitzen. Unter α) fällt das bekannte Zweibilder-Verfahren des R_3, unter β) die kotierte Projektion und ihre Verallgemeinerungen; unter γ) ordnet sich die Axonometrie ein, wobei Verfasser den sonst ausgeschloßenen Sonderfall des Pohlkeschen Satzes, wo die Bilder der 3 Achsen alle auf dieselbe Gerade fallen, ausführlich behandelt. Bei dem nächsten oben mit b) bezeichneten Hauptfall ist $n_z+n_\pi-(n-1)=m>0$. Dabei werden die Punkte auf Räume P_m abgebildet, sodaß man erst durch mehrerer solcher Abbildungen Punkt-Punkt-Korrespondenzen erhält. Ein Beispiel dafür ist die Abbildung der Punkte des R_3 aus 2 Geraden auf die Ebene Π, was mit Netzrißprojektion bezeichnet wird. Beim Hauptfall c) ist $n_z-n_\pi-(n-1)=-m<0$, und man bildet dann die P_m des P_n auf Punkte ab. Z.b. lassen sich die Geraden des R_3 aus einem Punkt so auf die Punkte einer festen Geraden projizieren, und 4 derartige Abbildungen stellen erst eine im allgemeinen eineindeutige Korrespondenz der Geraden auf die Punktequadrupel von 4 Geraden her. Durch Spezialisierungen folgen hieraus Zwei-Bild- und Zwei-Spur-Abbildungen des R_3. Die Spurabbildungen lassen sich aber auch erfassen, wenn man als Zentren die leere Menge zuläßt. *W. Burau.*

★ **Делоне, Б. Н. [Delone, B. N.] Элементарное доказательство непротиворечивости Планиметрии Лобачевского. [Elementary proof of absence of contradictions in the planimetry of Lobačevskiĭ.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 140 pp. 3.05 rubles.

In this book an effort is made to prove to readers with a secondary school education that the plane geometry of Lobačevskiĭ is logically as consistent as the plane Euclidean geometry taught in the schools. Incidentally the author hopes to provide the learned bodies of the land with a simple reference which they could offer their correspondents when acknowledging the receipt of a new, original, and irrefutable proof of Euclid's fifth postulate.

The author's main purpose is accomplished in the first two chapters of the book (pp. 25–63). In Ch. I he gives two sets of twenty postulates each: one for Euclidean and the other for Lobačevskian plane geometry. The first nineteen postulates are identical in the two sets: they are those of F. Schur (1856–1932) in which the undefined elements are point, line, motion. The twentieth postulates in the two sets are, respectively, the Euclidean and Lobačevskian parallel postulates.

In Ch. II affine transformations are studied, particularly those which leave invariant a semi-cone of revolution in which an element makes an angle of 45° with the axis of revolution. This leads the author to the Cayley-Klein Euclidean model of the Lobačevskian plane. Hence the conclusion that if the Lobačevskian plane geometry were inconsistent, the same would hold for the Euclidean case.

The chapter is written with consummate skill and great care. The author takes only for granted the rudiments of the mathematical equipment of those for whom the book is intended. Figures are abundant (27 in this chapter, about a hundred in all). Nothing is dismissed cryptically as being "perfectly obvious". And when the phrase occurs, it is followed by a suitable explanation. When the author needs the property of the equilateral hyperbola that the product of the distances of its points from the asymptotes is constant, he proves the proposition right there, in relation to the circumstances in which the proposition presents itself.

In the remainder of the book the author continues to study the geometry of the Lobačevskian plane. He considers the measure of an angle, length, areas, classification of motions, brings in Poincaré's model and Beltrami's pseudosphere.

The book opens with an excellent introduction on the history of non-Euclidean geometry which the author contributed originally to the journal "Priroda" (Nature) (1956) and closes with an appendix on "The connection between the geometry of Lobačevskiĭ and the special theory of relativity". *N. A. Court* (Norman, Okla.)

See also: Schwabhäuser, p. 863; Whitesitt, p. 869; McLachlan, p. 872; Blaschke, p. 922.

Convex Domains, Integral Geometry

Busemann, H.; and Petty, C. M. Problems on convex bodies. Math. Scand. 4 (1956), 88–94.

The authors formulate and comment on 10 problems concerning central-symmetric convex bodies in euclidean n-space. Let K be such a body, and denote by $K(u)$ its intersection with the hyperplane normal to the vector u and passing through the center. Furthermore, let $A(u)$ denote the $(n-1)$-dimensional volume of $K(u)$, and $C(u)$ the maximum of the n-dimensional volumes of the cones with base $K(u)$ and vertex in K. Most of the problems are concerned with inequalities relating these functions and the volume of K, and with characterizations of the ellipsoids in terms of them. Two examples: To find an estimate from above for the volume of K in terms of $A(u)$ such that the equality sign characterizes the ellipsoids. Are the ellipsoids characterized by the property that $C(u)$ is independent of u? The problems stem from, and their solutions would be of importance for, Minkowskian geometry [cf. H. Busemann, Comment. Math. Helv. 24 (1950), 156–187; MR 12, 527]. *W. Fenchel.*

Blaschke, Wilhelm. Zur Affingeometrie der Eilinien und Eiflächen. Math. Nachr. 15 (1956), 258–264.

Un ovale (curva piana convessa, chiusa) dicesi una linea P [P-Linie] se esso possiede una schiera semplicemente infinita di quadrangoli iscritti di area massima. Per una linea P si dà anzitutto la seguente caratterizzazione: un ovale è una linea P se per esso esistono coppie di diametri coniugati — cioè tali che le tangenti negli estremi di ciascuno di essi siano parallele all'altro — uno dei quali può essere scelto ad arbitrio. [Si intende per diametro di un ovale ogni segmento avente gli estremi sulla curva, e nei cui estremi le tangenti alla curva siano parallele]. Fra le linee P si considerano poi le quelle dotate di centro di simmetria, o linee R (R-Linien): per esse si ritrova un risultato di J. Radon [Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math. Phys. Kl. 68 (1916), 123–128]: un ovale è una linea R se e soltanto se esso è mutato in sè da una correlazione K, il cui quadrato è la simmetria rispetto a un punto (che è il centro dell' ovale).

Sin fanno poi applicazioni alle superficie chiuse convesse (ovaloidi) di S_3, tali che i cilindri ad esse circoscritti

abbiano sempre linee di contatto (separatrici di ombra: Selbstschattengrenzen) piane. Si ritrova anzitutto il risultato dell' Autore [ibid. **68** (1916), 50–55] secondo cui gli unici ovaloidi con separatrici d'ombra piane sono gli ellissoidi: si dimostra poi, di più, che ogni ovaloide, il cui contorno apparente con luce parallela su un piano sia una linea R, è un ellissoide. Si aggiungono alcune considerazioni relative al calcolo delle variazioni. — Come sempre nei lavori di Blaschke, i risultati sono ottenuti con metodo elementare ed estremamente elegante. *V. Dalla Volta.*

Blumenthal, Leonard M. Global subsets of the sphere. Arch. Math. **7** (1956), 367–373.

A subset G of the unit n-sphere S_n is called "global" on S_n if it is not contained in any hemisphere of S_n. Let $p_0, p_1, \cdots, p_n$ be $n+1$ points of S_{n-1} with no n-tuple in an S_{n-2}; then the $(n+1)$-tuple is global on S_{n-1} if and only if

$$\sum_0^n \Delta^{\frac{1}{2}}(p_0, p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_n)\cos pp_i=0$$

on S_n, where $\Delta(p_0, p_1, \cdots, p_n)$ denotes the determinant $|\cos p_ip_j|$. If a cartesian coordinate system in the E_{n+1} which contains S_n is introduced such that S_{n-1} has the equation $x_n=0$, then the equations

$$x_i'=(\cos pp_i)/[\sum_0^n \cos^2 pp_j]^{\frac{1}{2}} \quad (i=0, 1, \cdots, n)$$

define a mapping Φ which maps S_n (with north and south poles deleted) into a great S_{n-1}, say S_{n-1}^*. Some properties of this mapping are given. For instance, the equatorial hypersphere S_{n-1} is mapped congruently onto S_{n-1}^* if and only if the basic $(n+1)$-tuple is pseudo-equilateral (i.e. is either equilateral or becomes so upon reflecting one or more of its points in the center of S_{n-1}). *L. A. Santaló* (Buenos Aires).

Demaria, Davide Carlo. Sui ricoprimenti finiti della superficie sferica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 185–192.

B. Segre [Colloque sur les questions de réalité en géométrie, Liège, 1955, Thone, Liège, pp. 149–175; MR **17**, 994] has dealt with the following problem. Let S_r denote the r-dimensional unit sphere in euclidean $(r+1)$-space, and let n be a positive integer. For any covering of S_r by n closed sets, let μ be the maximum of the spherical diameters of the sets. Then find the minimum $\vartheta(n)$ of μ for all such coverings. In the case where $r=2$, which is the subject of this paper, Segre had found $\vartheta(n)$ for $n\leq 4$. The author determines $\vartheta(5)$, $\vartheta(6)$ and $\vartheta(8)$. The principal tool is the following lemma, which is of independent interest: If S_2 is covered by finitely many closed sets, then either three of the sets have a common point or one of the sets contains a pair of antipodal points. *W. Fenchel.*

See also: Ehrhart, p. 875; Marcus and Lopes, p. 877.

Differential Geometry

Plumier, S.; et Rozet, O. Sur les congruences de sphères de Ribaucour. Bull. Soc. Roy. Sci. Liège **25** (1956), 347–356.

This paper is concerned with an alternative method of determining congruences of spheres of Ribaucour given one of the enveloping surfaces of the spheres of the congruence. The method is explained and compared with the procedure used by Bianchi. Associated problems are discussed. A final example is discussed where the enveloping surface of the spheres of the congruence is the minimal surface of Enneper. *P. D. Thomas.*

Backes, Fernand. Recherches de géométrie anallagmatique. Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8° **29** (1956), no. 8, 36 pp.

This paper is concerned with certain surfaces W_a in three dimensional Euclidean space which are generalizations of the classical surfaces of Weingarten. At each point of a surface, consider the sphere tangent to the principal spheres of the surface at the point. A surface of Weingarten may be characterized by the property that the characteristic points of each sphere and the corresponding normal to the surface are coplanar. The author defines the surfaces W_a by the following generalization of this result: Consider a family F of circles each of which is normal to a given surface and which constitute a cyclic system. Let Σ be the sphere tangent to the two principal spheres of the surface at the points where these principal spheres are intersected by the corresponding circle of F. Then a surface W_a is defined by the property that each circle of F and the characteristic points of the corresponding sphere Σ are cospherical. An equivalent analytic condition. that a surface be W_a is given in terms of the differential element of angle between nearby spheres. The author then defines "W circles" by means of certain curvilinear nets on a surface. He shows that if the W circles constitute a cyclic system, then the surfaces orthogonal to the W circles are W_a surfaces. *A. Fialkow.*

Gorowara, K. K. General and hyper Darboux lines. Riv. Mat. Univ. Parma **6** (1955), 301–317.

L'auteur étudie, en utilisant le calcul tensoriel, des généralisations des lignes de Darboux de l'espace euclidien ordinaire [lignes de Darboux généralisées et hyper-lignes de Darboux]. Il indique quelques propriétés des lignes de Darboux généralisées et des hyper-lignes de Darboux, et montre comment certains résultats connus concernant les lignes de Darboux peuvent être déduits de résultats plus généraux relatifs à leurs extensions. *P. Vincensini.*

Guion, A. Sur un théorème de Dobriner. Bull. Soc. Roy. Sci. Liège **25** (1956), 400–404.

Partant d'une caractérisation des surfaces de l'espace euclidien ordinaire à lignes de courbure planes ou sphériques basée sur la considération des invariants de l'équation de Laplace associée au réseau de courbure, l'auteur étend aux surfaces dont les deux courbures moyenne et totale sont liées par une relation linéaire à coefficients constants, un théorème de Dobriner relatif aux surfaces à courbure totale constante. Pour ces dernières surfaces, si les lignes de courbure de l'un des systèmes sont planes, celles de l'autre système sont sphériques et tracées sur des sphères à centres alignés. Cette propriété subsiste, à l'alignement indiqué près, pour les surfaces à courbures moyenne et totale linéairement liées. *P. Vincensini.*

Saban, Giacomo. Formule integrali nelle deformazioni infinitesime di curve e superficie rigate chiuse. Arch. Math. **7** (1956), 380–383.

L'auteur étudie les déformations infinitésimales des courbes fermées tracées sur une sphère Σ de rayon 1 qui conservent les longueurs de ces courbes, les déformations s'effectuant sur la sphère Σ elle-même. Il arrive ainsi à des formules intégrales intéressantes, mettant en jeu de façon simple la courbure géodesique des courbes envi-

sagées. Il étend ensuite sa recherche aux déformations d'une surface réglée fermée K qui conservent sa longueur duale, et il montre comment, moyennant l'introduction des vecteurs duaux attachés aux génératrices rectilignes de K, on peut trouver des formules intégrales pour les variations des parties réelles et duales de la courbure sphérique duale de K. Une application intéressante des résultats obtenus est faite aux surfaces K développables.
P. Vincensini (Marseille).

Backes, F. Sur les congruences de cercles qui sont doublement stratifiables. Acad. Roy. Belg. Bull. Cl. Sci. (5) 42 (1956), 921–927.

Dans un travail antérieur [même Bull. (5) **40** (1954), 613–620; MR **16**, 69] l'auteur a étudié les couples de cercles (C, C') admettant des sphères focales, en correspondance telle que les périsphères se correspondent et que l'on puisse mener par chacun d'eux une infinité de sphères dont les points caractéristiques sont sur l'autre. Le travail actuel prolonge cette recherche, et traite du problème plus général de la recherche des congruences de cercles jouissant seulement de la dernière propriété énoncée, c'est-à-dire en correspondance biunivoque telle que l'on puisse, par chacun des cercles de l'une quelconque des deux congruences, mener une infinité de sphères dont les points caractéristiques sont sur le cercle correspondant de l'autre: les deux congruences de cercles constituent alors le couple doublement stratifiable le plus général. Le résultat est très remarquable. Les congruences des axes des cercles C et C' de deux telles congruences forment un couple de congruences rectilignes doublement stratifiable, les cercles C, C' sont orthogonaux à une sphère fixe (et possèdent par suite des sphères focales), en autre les cercles (Σ, Σ') [Σ et Σ' étant les sphères issues respectivement de C et C' et réalisant la stratification] sont W, leurs dix coordonnées vérifiant une même équation de Laplace. Réciproquement si (C) et (C') sont deux congruences de cercles orthogonaux à une sphère fixe, si deux sphères Σ_1, Σ_2 passant par C ont leurs points caractéristiques sur C' et deux sphères Σ_1', Σ_2' passant par C' ont les leurs sur C, et si les cercles (Σ_i, Σ_j') $(i, j=1, 2)$ sont W, les congruences (C), (C') sont doublement stratifiables. La démonstration de cette réciproque donnée par l'auteur montre que si l'on se donne le couple de congruences de droites doublement stratifiable le plus général, et si l'on construit les cercles ayant pour axes les droites de ce couple et orthogonaux à une sphère fixe quelconque, on obtient le couple de congruences de cercles doublement stratifiable le plus général.
P. Vincensini (Marseille).

Creangă, Ioan. Systèmes axials correspondants dans la correspondance par plans tangents parallèles entre deux surfaces. Rev. Univ. "Al. I. Cuza" Inst. Politehn. Iaşi 1 (1954), 9–21. (Romanian. Russian and French summaries)

An axial system of curves on a surface Σ, introduced by P. Sperry [Amer. J. Math. **40** (1918), 213–224], is defined as follows: In each point of Σ is given a straight line δ (axis) not lying in the tangent plane. A curve belongs to the axial system if the osculating plane in each of its points passes through δ. The axial system is thus a generalization of the geodesic lines, to which it reduces for δ normal to Σ.

Bompiani has shown that, given a point correspondence between two surfaces Σ_0 and Σ_1 which satisfies certain conditions of regularity and does not conserve any asymptotic line, there exists an axial system on Σ_0 which corresponds to an axial system on Σ_1. The axes δ_0 and δ_1 of these axial systems are called the axes of the correspondence.

The correspondence which the author considers in the present paper is that by parallel tangent planes (T_p). I.e. the point P_0 on Σ_0 corresponds to P_1 on Σ_1 if the tangent planes in P_0 to Σ_0 and P_1 to Σ_1 are parallel. With the help of Taylor expansions the author determines: 1) The 'Peterson' directions, i.e. the directions in P_0 which correspond to parallel directions in P_1. 2) The 'principal' directions of T_p i.e. those directions for which the modulus ds_1/ds_0 of linear expansion has extreme values. 3) The axes of the correspondence T_p as defined above.

Finally the following irregular cases are discussed: 1) P_0 or P_1 (or both) are parabolic. 2) The asymptotic lines correspond through T_p.
R. Blum.

Creangă, Ioan. Les surfaces dont les lignes cylindriques forment un système axial. Rev. Univ. "Al. I. Cuza" Inst. Politehn. Iaşi 2 (1955), 65–73. (Romanian. Russian and French summaries)

A line γ on a surface Σ is called cylindrical if the tangent planes to Σ along γ envelop a cylinder.

The differential equation of the cylindrical lines is of the second order and has the same form as the differential equation of an axial system [see the preceding review]. This leads the author to the problem of determining the surfaces whose cylindrical lines form an axial system. It is found that, with the obvious exception of the cylinders, the only surfaces with this property are the quadrics. Consequently, the only surfaces whose cylindrical lines coincide with the geodesic lines are the spheres (the cylinders excepted).
R. Blum (Saskatoon, Sask.).

Creangă, Ioan. Une propriété caractéristique des surfaces de 2e ordre. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) 1 (1955), 39–42. (Romanian. Russian and French summaries)

A tangential system of curves on a surface Σ, introduced by P. Sperry [Amer. J. Math. **40** (1918), 213–224], is defined by the property that the locus of the characteristic point of the tangent plane taken along an element of a curve of the system through a given point P of Σ is a straight line (lying in the tangent plane of P to Σ).

The following results have been found in the present note: 1) If on a given surface Σ one axial system [see previous two reviews] is in the same time a tangential system then every axial system on Σ is also a tangential system and inversely. 2) The only surfaces with this property are the nondegenerate quadrics. (The trivial case of the cones excepted).
R. Blum.

Poznyak, È. G. Approximation of infinitesimal deformations of surfaces of zero curvature. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 511–514. (Russian)

Let S be a sufficiently smooth piece of surface of the type of the 2-cell, which has zero curvature and is bounded by two rectilinear generators and two sufficiently smooth arcs. The author proves that the velocity field of an infinitesimal deformation of S can be approximated by velocity fields of infinitesimal deformations of prisms which approximate S. As applications of this method of approximation, the author gives five theorems concerning infinitesimal deformations of the "polygonal troughs" which he studied in earlier papers [same Dokl. (N.S.) **78** (1951), 205–207; Mat. Sb. N.S. **32(74)** (1953), 681–692; **12**, 857; **15**, 60].
L. C. Young (Madison, Wis.).

Vekua, I. N. Some questions concerning infinitesimal bending of surfaces. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 377–380. (Russian)

Jonas, Hans. Zur Theorie der Flächen mit Gewindekurven als Asymptotenlinien der einen oder der beiden Scharen. Math. Nachr. **15** (1956), 209–239.

Verfasser behandelt zunächst die charakteristische Bedingung $\sum \beta_1 \xi = 1$ zwischen den drei Integralen ξ, η, ζ der Moutardschen Differentialgleichung für eine Fläche mit den Asymptotenlinien α, β, deren Schar $\beta=$const von Gewindekurven gebildet wird. Dabei ist ξ, η, ζ der normierte Richtungsvektor der Flächennormale. Von einer vorgegebenen Fläche (x) mit Gewindekurven als Asymptotenlinien $\beta=$const kann man zu einer Regelfläche übergehen, deren Geraden den letzteren entsprechen. Die allgemeinsten Flächen dieser Art gehen aus Regelflächen durch asymptotische Transformation hervor. — Weiterhin werden Flächen mit Gewindekurven als Asymptotenlinien beider Scharen behandelt, darunter an erster Stelle die Terracinischen Flächen, die aus Regelflächen zweiten Grades durch nicht spezielle asymptotische Transformationen hervorgehen. Für diese gibt es quadraturfreie Darstellungen, die ohne Verwendung der Lösung der Moutardschen Gleichung erhalten werden. Dabei gewinnt Verfasser auch die bereits bekannten charakteristischen Bedingungen für eine Fläche mit Gewindekurven als Asymptotenlinien beider Scharen, die über die Ergebnisse von E. P. Lane und MacQueen hinaus auf eine bemerkenswerte Funktionalgleichung zurückgeführt werden [Amer. J. Math. **40** (1938), 337–344]. In diese allgemeinere Flächenklasse ordnet sich die von K. Strubecker behandelte Familie der Φ_1-Flächen ein, deren Darstellung und Eigenschaften Verfasser einen weiteren Abschnitt seiner Untersuchungen widmet. Die Φ_1-Flächen lassen sich als asymptotische Transformierte der imaginären Quadrik $x^2+y^2+z^2+1=0$ auffassen und sind demnach Terracinische Flächen. Ein weiterer Satz lautet: Das polarreziproke Flächenpaar (x), (x') der Klasse Φ_1 mit Gewindekurven als Asymptotenlinien beider Scharen erhält man aus den Biegungsflächen vom Typus

$$\sum d\bar{x}^2 = (1-u^2)du^2 - 2uvdudv + (1-v^2)dv^2$$

des Weingartenschen Rotationsparaboloides, und zwar dargestellt durch die bei den Verbiegungen starr gekoppelten tangentialen Vektoren

$$x = -\frac{\bar{x}_u}{u}, \quad x' = -\frac{\bar{x}_v}{v}.$$

Die Φ_1-Flächen sind aequivalent mit den Flächen der absoluten Krümmung Null im elliptischen Raum. Der die Flächen Φ_1 behandelnde Hauptteil der Untersuchung wird mit der Konstruktion gewisser einfacher Regelflächen, mit denen die Flächen Φ_1 durch asymptotische Transformation zusammenhängen beschlossen. — Es folgt noch eine Behandlung der Tetraedralfläche

$$x^{2/3} + y^{2/3} - z^{2/3} = 1,$$

auf der die Asymptotenlinien beider Scharen aus reellen kubischen Raumkurven bestehen. *M. Pinl* (Köln).

Löbell, Frank. Die Integrabilitätsbedingung für Ortsfunktionen bei nichtintegrablen Bezugsystemen. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1956**, 33–39 (1957).

In Euclidean three-space let there be given three mutually orthogonal congruences of curves with arc lengths s_1, s_2, and s_3 respectively. If f is a function of position, it is shown that

$$\frac{\partial}{\partial s_2}\frac{\partial f}{\partial s_1} - \frac{\partial}{\partial s_1}\frac{\partial f}{\partial s_2} = G_1\frac{\partial f}{\partial s_1} + G_2\frac{\partial f}{\partial s_2} - 2A_3\frac{\partial f}{\partial s_3},$$

where G_1 and G_2 are the lateral curvatures of the first two congruences and A_3 is the mean Schränkung of the third congruence. The Schränkung of a congruence is the reciprocal of the parameter of distribution. The paper is written from the point of view of the "Natural Geometry" of E. Cesàro [Lezioni di geometria intrinseca, Napoli, 1896]. *C. B. Allendoerfer* (Seattle, Wash.).

Denis, F. Sur certaines congruences de droites. Bull. Soc. Roy. Sci. Liège **25** (1956), 383–386.

$g(u, v)$ représentant une congruence de droites rapportée à ses développables u, v, et pour laquelle les foyers correspondant aux variations respectives de u et v sont y et z, M. Chem a envisagé une famille de telles congruences définies par l'annulation d'un certain invariant [Tôhoku Math. J. **40** (1935), 293–316], et il a caractérisé les congruences en question par le fait que la congruence bilinéaire commune aux complexes d'accompagnement de Waelsh relatifs au foyer z (y) de g et au foyer correspondant de la génératrice infiniment voisine obtenue par variation infinitésimale de v (u) a ses directrices confondues.

L'auteur donne ici une nouvelle caracterisation géométrique des congruences du type envisagé, consistant en ce que, sur la surface G image de l'une quelconque de ces congruences sur l'hyperquadrique de Klein, les courbes $du=v$ $(dv=0)$ constituent l'une des cinq familles de lignes principales. Ces courbes sont d'ailleurs des lignes quasi-asymptotiques de G, et le point z est foyer inflexionnal triple pour la complexe engendré par les droites du faisceau plan de sommet z situées dans le plan focal correspondant lorsque u et v varient. *P. Vincensini.*

Terracini, Alessandro. I sistemi infiniti di piani nello spazio a cinque dimensioni. Univ. e Politec. Torino. Rend. Sem. Mat. **15** (1955–56), 75–104.

Ce travail est la traduction Italienne de conférences faites par l'auteur à la Faculté des Sciences de Marseille en Mai 1956, dans lesquelles il reprend d'un point de vue plus didactique, et en enrichissant le sujet de vues nouvelles et de prolongements importants, la matière d'une conférence faite par lui [Colloque de Géométrie Différentielle, Louvain, 1951, Thone, Liège, 1951, pp. 51–65; MR 13, 490]. L'auteur commence par donner un sens précis à la notion des systèmes simplement infinis de plans de l'espace S_5 à cinq dimensions [systèmes Δ], tels qu'il y ait incidence de chaque plan du système et du plan infiniment voisin suivant un certain ordre (σ) d'approximation. Cette notion d'ordre d'approximation constitue un très important perfectionnement d'une notion analogue introduite par C. Segre [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) 30 (1921), 1° semestre, 227–231], à savoir la notion "d'ordre de voisinage", perfectionnement grâce auquel s'est trouvé ouvert un nouveau champ de recherches du plus haut intérêt.

L'ordre d'approximation peut prendre, effectivement, les valeurs 2, 4, 6, 8, 10, 12, 14, 16, ∞, cette dernière valeur correspondant au cas où dans le système Δ deux plans quelconques seraient toujours incidents. L'introduction des coordonnées de Grassmann des plans de S_5 et leur interprétation sur la variété V_9 (de Grassmann) dont les points représentent, dans l'espace projectif S_{19}, les

plans de S_5, permet de caractériser géométriquement ces différents ordres d'approximation, et de faire une analyse minutieuse des circonstances qui se présentent lorsque σ prend toutes les valeurs indiquées. La notion d'ordre d'approximation éclaire d'un jour nouveau, et permet de pénetrer jusque dans leurs plus intimes fondements, de nombreuses questions de géométrie différentielle. A cette fin l'auteur a tiré un très heureux parti de l'introduction de ce qu'il appelle des "bifaisceaux", êtres géométriques constitués par une génératrice a de la quadrique de Klein, et par deux faisceaux plans de droites contenant chacun la génératrice en question, situés dans l'espace linéaire a_3 tangent à la quadrique de Klein le long de a et polaires réciproques par rapport à cette quadrique. Chacun de ces bifaiceaux est l'image d'une calotte de l'espace ordinaire; les ensembles e_i de dimension i de calottes de l'espace ordinaire ont pour images des ensembles W_i de bifaisceaux, et l'examen des particularités que peuvent présenter ces différents W_i permet une étude très suggestive des configurations e_i les plus intéressantes au point de vue géométrique. Les deux invariants I_1 et I_2 d'un système de deux calottes reçoivent ainsi des significations géométriques remarquables, et la considération des différentes particularités que peuvent présenter les W_1 en relation avec la quadrique de Klein, conduit à une étude extrêmement élégante des possibilités de réalisation des divers systèmes Δ correspondant aux diverses valeurs possibles de l'ordre σ d'approximation. La notion d'ordre d'approximation peut être étendue aux systèmes de plans de S_5 de dimension deux. L'auteur, prenant comme point de départ le cas $\sigma\geqq 2$, donne des indications sur ce nouveau champ de recherches, et montre en particulier le rôle qu'y jouent les congruences quadratiques et les congruences W de S_5, ainsi que la notion de déformation projective du troisième ordre d'une surface. En terminant son exposé l'auteur attire l'attention sur une généralisation des systèmes Δ de S_5 dont il s'est occupé dans un Mémoire antérieur [Scritti Matematici offerti a Luigi Berzolari, Pavia, 1936, pp. 449–478], ainsi que sur l'intérêt qui s'attache à l'extension de la notion d'ordre d'approximation à des ensembles d'êtres mathématiques définis par une relation déterminé entre deux éléments infiniment voisins de l'ensemble.

P. Vincensini (Marseille).

Vagner, V. V. Algebraic theory of tangent spaces of higher orders. Trudy Sem. Vektor. Tenzor. Anal. **10** (1956), 31–88. (Russian)

Die vorliegende Arbeit bringt entscheidende Algebraisierungen zur Theorie der Berührungsräume allgemeiner differentialgeometrischer Räume. Die meisten Teile derselben haben daher zunächst einen stark algebraischen Charakter. Der wesentliche Grundbegriff ist der einer kommutativen Algebra $W_{(v,n)}$ über dem reellen Körper von der sog. Nilpotenzstufe $v+1$. Das bedeutet: $W_{(v,n)}$ besitzt die Dimension $\binom{n+v}{n}-1$, und als Basis nehme man zunächst n Vektoren (die sog. Pseudobasis), und dann alle Produkte bis zu v Faktoren zwischen diesen, während alle Produkte von mehr als v Faktoren beim Multiplizieren verschwinden sollen. Die Elemente der Algebra $W_{(v,n)}$ kann man ersichtlich auf die Polynome der Algebra $Q_v[x_1, \cdots, x_n]$ eineindeutig abbilden, wobei in Q_v alle Glieder von höherem Grade als v beim Multiplizieren verschwinden sollen. Eine derartige Abbildung bedeutet die Einführung eines Koordinatensystems in $W_{(v,n)}$. Die Automorphismen von $W_{(v,n)}$ bilden die sog. Differentialgruppe $D_{(v,n)}$. Jede Unteralgebra von $W_{(v,n)}$ mit derselben Nilpotenzstufe heißt regulär, und alle derartigen regulären Unteralgebren der gleichen Pseudodimension m $(\leqq n)$ erweisen sich als automorph innerhalb von $W_{(v,n)}$. Ferner ist jede reguläre Unteralgebra $W_{(v,m)}\subset W_{(v,n)}$ in einem kleinsten Ideal $\mathfrak{J}_{(v,m)}$ vom sog. genus m eindeutig enthalten. Nach solchen Idealen lassen sich wiederum Faktoralgebren $W_{(v,n-m)}$ bilden. Auch zwischen $W_{(v,n)}$ und $W_{(u,n)}$ $(u\leqq v)$ besteht ein Homomorphismus, bei dem $W_{(v,n)}$ isomorph einer Faktoralgebra $W_{(v,n)}/W_{(v,n)}^{u+1}$ nach einem Ideal $W_{(v,n)}^{u+1}$ aus $W_{(v,n)}$ ist. Die Restklassen nach diesem Ideal bleiben bei allen Automorphismen von $W_{(v,n)}$ invariant. Schließlich wird in § 3 noch gezeigt, daß eine Differentiation in $W_{(v,n)}$, die den üblichen Produktregeln genügt, dadurch bestimmt ist, daß man den Vektoren der Pseudobasis gewisse andere Grössen der Algebra willkürlich zuordnet.

Der folgende § 4 bringt dann zunächst einige Betrachtungen über kommutative Halbgruppen, in denen nur eine assoziative Verknüpfung erklärt ist, die additiv geschrieben wird. In einem solchen System S kann man in gewissen Fällen $-s$ als dasjenige, wenn überhaupt, dann eindeutig vorhandene Element s' erklären, das $2s+s'=s$ und $2s'+s=s'$ erfüllt. Dies führt dazu, durch $s_1\prec s_2$ bei $s_1-s_2=s_1-s_1$ eine Ordnung zwischen gewißen Elementen von S einzuführen. Diese Teilordnung ist transitiv, und aus $s_1\prec s_2$ folgt $ms_1\prec ms_2$ für beliebige positive und negative ganze m und auch für reelle m, wenn diese als zulässiger Operatorenbereich S zu einer verallgemeinerten Vektoralgebra machen. An Stelle der 0 treten die sog. stabilen Elemente s_0 mit der Eigenschaft $ms_0=s_0$ für alle m. Bei Vorhandensein einer Multiplikation in S, die aber weder assoziativ noch kommutativ zu sein braucht, heißt S eine verallgemeinerte Algebra. Solche Halbgruppen lassen sich auf gewöhnliche Gruppen homomorph abbilden. Hierbei treten in S als Urbilder des Nullelementes sog. verallgemeinerte Normalteiler auf, die alle stabilen Elemente enthalten. Analoges gilt, wenn S ein verallgemeinerter Vektorraum bzw. eine verallgemeinerte Algebra ist. Ein Beispiel für diese Begriffsbildungen hat man in der Menge F derjenigen Funktionen, die auf Teilmengen einer Menge M erklärt sind und als Werte Elemente einer anderen Menge A, die meist als abelsche Gruppe anzunehmen ist, haben. Die Verknüpfung $f_1\mathsf{T}f_2$ zweier Funktionen wird dabei durch $(f_1\mathsf{T}f_2)(m)=f_1(m)\mathsf{T}f_2(m)$ erklärt, wobei rechts unter T die im Wertebereich A schon vorhandene Verknüpfung zu verstehen ist. Falls f_1 und f_2 gar keinen gemeinsamen Definitionsbereich auf M besitzen, ist unter $f_1\mathsf{T}f_2$ die leere Funktion zu verstehen. Die Menge F kann offenbar auch als Menge von Teilmengen im Produkt $M\times A$ aufgefasst werden; sie ist eine abelsche Halbgruppe, wobei diejenigen Funktionen, die nur das Nullelement aus A zum Wertbereich haben, die Rolle der stabilen Elemente spielen. Es gilt $f_1\prec f_2$, wenn die zugehörigen Bildmengen aus $M\times A$ ineinander enthalten sind.

Im § 5 stellt sich dann die Menge $F_{(v,n,\xi_0)}$ aller in einem Punkt ξ_0 v-mal differenzierbaren Funktionen von n reellen Variabeln als verallgemeinerte Algebra heraus, die auf die Algebra $P_v[x_1, \cdots, x_n]$ der zugehörigen Taylorpolynome höchstens v-ten Grades homomorph abgebildet wird. Der Kern dieses Homomorphismus ist das Normalideal $\mathfrak{B}_{(v,n,\xi_0)}$ aus $F_{(v,n,\xi_0)}$. $\mathfrak{B}$ ist die Menge aller mindestens von der Ordnung $v+1$ in ξ_0 verschwindenden Funktionen. Jede Teilabbildung zwischen den affinen Räumen R_n und R_m, die durch v-Mal differenzierbare Funktionen vermittelt wird, und wobei die Punkte ξ_0 und η_0 zugeordnet sein sollen, definiert einen Homomorphismus zwischen

$F_{(n,\xi_v)}$ und $F_{(m,\eta)}$, wobei die Ideale $\mathfrak{B}_{(v,n,\xi_v)}$ und $\mathfrak{B}_{(v,m,\eta_v)}$ sich entsprechen.

Erst im § 6 wird dann wieder der Anschluß an die Differentialgeometrie gewonnen. C_n sei ein Veblen-Whiteheadscher Raum (kurz V-W-Raum genannt) von der Differenzierbarkeitsklasse r. Mit $F_{(C_n,c_0)}$ wird die verallgemeinerte Algebra aller im Punkt $c_0 \epsilon C_n$ stetigen skalaren Funktionen und mit $F_{(v,C_n,c_0)}$ die der dort v-Mal differenzierbaren Funktionen bezeichnet. Beide Begriffe erweisen sich als unabhängig von der speziellen Wahl des Koordinatensystems in der Umgebung von c_0. Wesentlich sind ferner die lokalen Algebren $W_{(v,n)}(c_0)$; sie heißen die kovarianten Berührungsräume v-ter Stufe in c_0. Als kontravarianter Berührungsraum der Stufe v wird die Menge $T_{(v,n)}(c_0)$ aller Homomorphismen der Algebra $W_{(v,n)}(c_0)$ auf die Algebra $Q_v[x]$ verstanden. Dabei kann v zwischen 1 und r variieren. Alle früher eingeführten Homomorphismen usw. gewinnen jetzt ihre Bedeutung für die Differentialgeometrie von C_n. Man kann z.B. die kovarianten Berührungsräume gleicher Stufe v, die zu verschiedenen Punkten c_0 und c_0' desselben oder verschiedener Räume C_n gehören, dadurch aufeinander beziehen, daß man sie beide auf die Algebra $Q_v[x]$ homomorph abbildet. Bei Unterräumen $C_m \subset C_n$, die ihrerseits wieder V-W-Räume sind, kann sinngemäß zwischen einer äußeren und inneren Differentiationsstufe unterschieden werden. Ist c_0 ein gemeinsamer Punkt von C_m und C_n, so läßt sich von den Idealen $W_{(v,n)}(c_0)$ und $W_{(v,m)}(c_0)$ sprechen, die je zu C_n und C_m gehören. Zwischen diesen Idealen existiert ein Homomorphismus, der das reguläre Ideal $\mathfrak{J}_{(v,n-m)}(c_0)$ aus $W_{(v,n)}(c_0)$ als Kern besitzt. Hiermit läßt sich der wichtige Begriff der Berührung bestimmter Ordnung in folgender Weise definieren: Die Teilräume C_1' und C_m' aus C_m $(1 \leq m)$ berühren sich in einem gemeinsamen Punkte c_0 in der Ordnung v, wenn für die oben definierten Ideale gilt: $\mathfrak{J}_{(v,n-m)}(c_0) \subset \mathfrak{J}_{(v,n-1)}(c_0)$.

W. Burau (Hamburg).

Chern, Shiing-shen; and Lashof, Richard K. On the total curvature of immersed manifolds. Amer. J. Math. **79** (1957), 306–318.

Let M^n be an oriented differentiable C^∞ manifold immersed in a Euclidean E^{n+N}, where $N \geqq 1$. The bundle B_ν of $(N-1)$ dimensional spheres on M^n is a manifold of dimension $n+N-1$ which has been treated, among others, by Lipschitz, Killing, Weyl, Fenchel, and Allendoerfer. Locally, it can be taken to be a hypersphere in E^{n+N}, and hence has an ordinary Gaussian curvature $G(p, \nu)$, where $p \epsilon M^n$ and $\nu = \nu(p)$ is a unit normal to M^n at p. Let

$$K^*(p) = \int |G(p, \nu)| d\sigma_{N-1},$$

where the integration is over the sphere of unit normals at $p \epsilon M^n$. Then the total curvature of M^n is defined to be $K = \int_{M^n} K^*(p) dV$ if this converges.

Three theorems are proved for the case where M^n is compact: (1) $K \geqq 2c_{n+N-1}$, where c_{n+N-1} is the area of a unit hypersphere in E^{n+N}. (2) If $K < 3c_{n+N-1}$, then M^n is homeomorphic to an n-sphere. (3) If $K = 2c_{n+N-1}$, then M^n belongs to a linear subvariety E^{n+1} and is imbedded as a convex hypersurface in E^{n+1}. The converse of this is also true.

C. B. Allendoerfer (Seattle, Wash.).

Laugwitz, Detlef. Zur Differentialgeometrie der Hyperflächen in Vektorräumen und zur affingeometrischen Deutung der Theorie der Finsler-Räume. Math. Z. **67** (1957), 63–74.

The author begins with a hypersurface in an affine space and using osculating quadratic surfaces defines a ds^2 for the hypersurface. He imposes analytic conditions on the surface to assure that the ds^2 has the correct properties. He then compares his results with those of Salkowski [Affine Differentialgeometrie, de Gruyter, Berlin-Leipzig, 1934] and shows that the affine fundamental metric of Salkowski and the one induced by the above method differ only in sign. The author also introduces a cubic fundamental form and proves that the quadric and the cubic forms under certain restrictive conditions determine one and only one surface up to a homogeneous affine transformation. The paper closes with an affine geometric interpretation of É. Cartan's study of Finsler spaces [Les espaces de Finsler, Hermann, Paris, 1934].

L. Auslander (Bloomington, Ind.).

Arghiriade, E. Sur les quadriques de Davis-Gambier. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **8** (1956), 617–630. (Romanian. Russian and French summaries)

Les quadriques ayant au point P d'une surface S un contact du deuxième ordre, coupent S suivant une courbe, qui a en P un point triple. Soient Q_1 et Q_2 les quadriques osculatrices asymptotiques il existe un faisceau de quadriques (de Davis-Gambier) se raccordants avec Q_1 et Q_2 le long des tangentes asymptotiques. L'auteur démontre quelques théorèmes sur le cas [considéré par Green, Amer. J. Math. **60** (1938), 649–666, p. 653] que les quadriques sont axiales. Relations avec les quadriques de Darboux et de Wilczynski-Bompiani.

O. Bottema (Delft).

Nicolescu, Al. Quelques propriétés des surfaces de Tzitzéica. Com. Acad. R. P. Romîne **6** (1956), 1065–1071. (Romanian. Russian and French summaries)

Tzitzéica [C. R. Acad. Sci. Paris **180** (1925), 1715–1716; Ann. Sci. École Norm. Sup. (3) **42** (1925), 379–390= Oeuvres, T. I, Acad. Roumaine, Bucarest, 1941, pp. 380–393] a considéré les surfaces qui appartient aux groupes centre-affine et affine-parabolique. L'auteur démontre quelques propriétés nouvelles, dont nous citons: les surfaces dont les normales affines demeurent parallèles à une direction fixe sont applicables affines sur un plan.

O. Bottema (Delft).

Decuyper, Marcel. Sur quelques couples de surfaces ayant mêmes premiers axes relativement au réseau conjugué commun. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 1018–1027.

If two analytic surfaces (M_1), (M_3) are related by a one-to-one correspondence between their points $M_1(u, v)$, $M_3(u, v)$, a conjugate net $N_1(u, v)$ exists on (M_1) which corresponds to a conjugate net $N_3(u, v)$ on the other. The author has shown that a pair of surfaces (M_1), (M_3) whose corresponding conjugate nets $N_1(u, v)$, $N_3(u, v)$ have a common axis congruence depends on ten arbitrary functions of a parameter. He treats two particular cases which correspond to inverse, or direct, intersection of the conjugate nets. The two conjugate nets $N_1(u, v)$, $N_3(u, v)$ are said to intersect inversely, or directly, if the (u, v)-tangents to N_1 at M_1 intersect the (v, u)-tangents, or (u, v)-tangents, to N_3 at M_3, respectively. The following results were obtained: In order that the conjugate nets $N_1(u, v)$, $N_3(u, v)$ intersect inversely ("se coupent inversement") it is necessary and sufficient that they be generated by two opposite vertices of a periodic sequence of Laplace of period four. In order that the conjugate nets intersect directly, it is necessary that the curves of the

two nets be plane curves. This last result is known; for the curves of the conjugate nets $N_1(u, v)$, $N_3(u, v)$ coincide with the axis curves on (M_1) and (M_3), respectively, when the conjugate nets N_1, N_3 intersect directly. And this coincidence implies that the curves of the nets N_1, N_3 are plane curves [cf. E. P. Lane, A treatise on projective differential geometry, Univ. of Chicago Press, 1942, p. 262; MR 4, 114]. *P. O. Bell.*

★ **Ţiţeica, Gheorghe. Geometrie diferenţială proiectivă a reţelelór. [Differential projective geometry of nets.]** Editura Academiei Republicii Populare Romîne, Bucureşti, 1956. 286 pp. 18.10 lei.

Translation of Géométrie différentielle projective des réseaux [Gauthier-Villars, Paris, 1924]. *R. Blum.*

Vaona, Guido. Deformazione proiettiva di uno strato di superficie dello spazio ordinario. Atti Sem. Mat. Fis. Univ. Modena 7 (1953–54), 28–67 (1956).

L'auteur commence par rappeler la définition dûe à E. Čech [Convengo Internazionale di Geometria Differenziale, Italia, 1953, Edizioni Cremonese, Roma, 1954, pp. 266–273; MR 16, 168] de la déformation projective d'une famille d'hypersurfaces telles que par un point de l'espace (ou de la région que l'on considère) passe une hypersurface et une seule de la famille [la famille est alors dite un "strato", ce que nous traduirons par un „clivé"], puis étudie un certain nombre de questions relatives aux transformations ponctuelles entre deux espaces ordinaires qui sont des transformations projectives pour un clivé de surfaces développables. La mise en équations du problème de la déformation projective d'un clivé de surfaces non développables fournit, après des normalisations convenables, deux systèmes de trois équations aux dérivées partielles linéaires homogènes du 2ème ordre, conduisant à la considération de deux types de déformations pour lesquels les surfaces de clivage sont respectivement R et R_0. L'auteur étudie séparément les déformations projectives des clivés des deux types obtenus, établissant pour chacun d'eux de nombreuses propriétés relatives soit aux clivés eux-mêmes soit à leurs déformations, et portant notamment l'attention sur ceux pour lesquels le réseau de déformation projective de chaque surface est le réseau R ou R_0 correspondant. Le cas des clivés dont les surfaces possèdent des réseaux R formés de courbes planes est étudié d'une façon particulière. Les surfaces de clivage sont alors telles que leurs réseaux R sont formés de courbes planes dont les plans passent par l'une ou l'autre de deux droites fixes. Ces surfaces et leur déformation ont été étudiées de manière approfondie par B. Segre [Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 2 (1931), 49–189] qui en a fait connaître de nombreuses propriétés géometriques, certaines desquelles ont permis à l'auteur du travail actuel d'énoncer des constructions élégantes et remarquablement simples des clivés formés de surfaces R du type envisagé et de leurs déformés projectifs. Comme cas particulier important de clivés de surfaces R_0, l'auteur considère ceux pour lesquels sur chaque surface de clivage le réseau de déformation projective est le réseau asymptotique. Les formules de déformation peuvent alors recevoir une forme canonique remarquable à laquelle s'attachent de très intéressantes remarques géométriques. *P. Vincensini* (Marseille).

Terracini, Alessandro. Particolari superficie W dello S_5 in relazione con le loro linee principali. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 90 (1955–56), 591–603.

Une surface F de l'espace projectif à cinq dimensions S_5 admet cinq familles ∞^1 de lignes principales, c'est-à-dire, en utilisant la notion de plans incidents suivant un ordre d'approximation donné [voir A. Terracini, Scritti Matematici offerti a Luigi Berzolari, Paria, 1936, pp. 449–478], telles que, dans le système formé par les plans tangents à F le long d'une quelconque de ces lignes, deux plans tangents infiniment voisins soient incidents suivant un ordre d'approximation $\sigma \geqq 4$. Envisagement plus spécialement les surfaces [non représentant des équations de Laplace et non de Véronèse] pour lesquelles, les coefficients des systèmes d'équations linéaires et homogènes du 3ème ordre vérifiés par les coordonnées courantes exprimées au moyen de deux paramètres convenables u, v sont constants [surfaces W], l'auteur étudie les surfaces pour lesquelles l'un au moins des cinq systèmes de lignes principales présente le cas $\sigma=\infty$ pour l'ordre d'approximation précédemment défini. Cela peut arriver, soit parce que les plans tangents le long d'une ligne principale quelconque d'un système sont contenus dans un S_4 [cas a), admettant un sous-cas a') pour lequel les lignes principales d'un système sont dans des S_3 passant par un même plan fixe π (lignes principales coniques)], soit parce que les lignes principales d'un système sont planes [cas b)], soit enfin parce que les plans tangents le long d'une ligne principale d'un système appartiennent à l'une des familles de plans d'une hyperquadrique $V_4{}^2$ non singulière, ou bien sont tangents à une surface de Véronèse ou sont des plans de coniques d'une telle surface [cas respectifs c), d), e)]. Et la considération de ces différents cas conduit aux résultats suivants, où la notation $(ma, nb, \cdots)$, $[m, n=1, 2, \cdots, 5, m+n=5]$, indique les surfaces pour lesquelles n systèmes de lignes principales présentent le cas a) et n le cas b): Il n'existe aucune surface W admettant cinq systèmes distincts de lignes principales appartenant au type (5a). Il existe des surfaces W admettant quatre systèmes de lignes principales du type a); ces surfaces sont alors du type (4a, c), et plus précisément du type (4a', c), et elles sont toutes homographiques entre elles. Il n'existe pas de surfaces W avec cinq systèmes distincts de lignes principales du type (3a, 2b); tandis qu'il en existe du type (2a, 3b), ces dernières, toutes homographiques entre elles, possédant trois systèmes de lignes principales planes. Dans tous les cas d'existence signalés, l'auteur donne les représentations paramétriques explicites des surfaces envisagées. *P. Vincensini* (Marseille).

Götz, H. Zur konformen Kurventheorie. Monatsh. Math. 60 (1956), 205–211.

L'Autore considera le curve in uno spazio conforme C_3 (le curve sono supposte di classe $\geqq 5$), servendosi delle coordinate pentasferiche di E. Cartan; ad evitare gli usuali inconvenienti dovuti all'omogeneità delle coordinate, l'A. impiega una semidifferenziozane invariante, nell'ordine di idee di G. Bol [Math. Z. 54 (1951), 141–159; MR 13, 160]. Benchè lo scopo principale del lavoro sia lo studio delle curve, vengono date anche formule valide più in generale, per poterle applicare — in un futuro lavoro — anche alle superficie di C_3. Viene poi data la teoria conforme delle curve, e si determinano due invarianti conformi (curvature e torsione), che si mettono in relazione a risultati di altri autori [van der Woude, Nederl. Akad. Wetensch. Proc. 10 (1948), 16–24; MR 9, 467; Maeda: Jap. J. Math. 16 (1940), 177–232; MR 2, 156]. Infine si interpretano geometricamente i risultati trovati. *V. Dalla Volta* (Roma).

Hodova, R. N. Infinitesimal classification of curves of order two in the Lobačevskiĭ plane. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki. 10 (1956), 45–54. (Russian)

Villa, M.; e Muracchini, L. Sulle corrispondenze fra superficie della varietà di Segre. Rev. Un. Mat. Argentina 17 (1955), 329–334 (1956).

Sia γ una corrispondenza puntuale fra due superficie Σ, $\bar{\Sigma}$ di due spazi proiettivi S_r, $\bar{S}_r$; dicesi che un'omografia ω fra gli spazi S_r, $\bar{S}_r$ è tangente a γ di ordine k (>0) in una coppia P, $\bar{P}$ di punti corrispondenti e appartenenti rispettivamente a $\bar{\Sigma}$, Σ, se ogni curva $C \subset \Sigma$ per P viene trasformata da γ e da ω in due curve $\bar{C}$, $\bar{C}_0$ ($\subset \Sigma$) per $\bar{P}$, aventi ivi un contatto analitico di ordine $\geqq k$, il segno $>$ valendo solo per particolari direzioni uscenti da P, che diconsi caratteristiche. Gli autori applicano tale concetto al caso in cui le superficie considerate, siano esse Φ, $\bar{\Phi}$, appartengano a due varietà di Segre V_4, $\bar{V}_4$ di due spazi proiettivi S_8, $\bar{S}_8$, e considerano le omografie I fra S_8, $\bar{S}_8$ che mutano V_4 in $\bar{V}_4$, trasformando una delle schiere di piani della prima varietà di Segre in una prefissata della seconda. Ciò posto, gli AA. dimostrano il seguente risultato:

Una corrispondenza γ fra due superficie Φ, $\bar{\Phi}$, appartenenti rispettivamente a V_4, $\bar{V}_4$ possiede in una coppia P, $\bar{P}$ di punti corrispondenti generici al più tre coppie di direzioni caratteristiche relativamente a qualche omografia I. Una omografia di I tangente a γ in P, $\bar{P}$, dà luogo al più a due coppie di direzioni caratteristiche. Le coppie nominate possono diventare infinite se γ muta le tangenti alle tre quasi-asintotiche nelle analoghe rette per $\bar{P}$. *V. Dalla Volta.* (Roma).

Cantoni, Lionello. Sulle corrispondenze linearizzanti e sui riferimenti intrinseci in una coppia a jacobiano nullo. Boll. Un. Mat. Ital. (3) 11 (1956), 402–411.

Sia T una trasformazione puntuale tra due spazi proiettivi S_r, S_r' ($r>2$), e siano O, O' due punti corrispondenti, risp. di S_r, S_r'. L'A. determina un riferimento intrinseco in O, O', sia nel caso che la coppia considerata sia regolare, sia in quelche O, O' sia una coppia a Jacobiano nullo. *V. Dalla Volta* (Roma).

See also: Freudenthal, p. 871; Sumitomo, p. 931; Mutō, p. 932; Vyčichlo, p. 963; Chovitz, p. 978.

Riemannian Geometry, Connections

Mautner, F. I. Geodesic flows on symmetric Riemann spaces. Ann. of Math. (2) 65 (1957), 416–431.

Let M be a Riemann manifold, p a point on M and b a unit tangent vector to M at p. For any real t, consider the geodesic through p in the positive direction of b and denote by p_t and b_t, respectively, the point on the geodesic at distance t from p and the unit tangent vector at p_t in the positive direction of the geodesic. The mappings T_t: $b \rightarrow b_t$ ($-\infty < t < +\infty$) then define a one-parameter transformation group of the manifold $B(M)$ of all unit tangent vectors at all points of M, called the geodesic flow on the Riemann manifold M. The T_t are measure-preserving transformations with respect to a natural measure μ on $B(M)$.

Now, let G be a connected non-compact semi-simple Lie group with finite center, K a maximal compact subgroup of G and G/K the homogeneous space of cosets gK, $g \in G$. A discrete subgroup Γ of G then acts properly discontinuously on G/K and, if this transformation group has no fixed point, a real analytic manifold $M = \Gamma \backslash G/K$ can be obtained from G/K by identification modulo Γ; the points of M are given by double cosets ΓgK in G. Furthermore, M is a Riemann manifold with respect to the Riemann metric induced on M by that of G/K. In the present paper, the author studies the geodesic flow on such a Riemann manifold $M = \Gamma \backslash G/K$, and the main results are as follows: Let E_1 be the unit sphere of the tangent space to G/K at K. As $B(M)$ may be canonically identified with the factor manifold $\Gamma \backslash B(G/K)$, every element of $B(M)$ can be written in the form Γge with $g \in G$, $e \in E_1$. The author then proves that the geodesic flow on $B(M)$ is given in the form

$$T_t\colon \Gamma ge \rightarrow \Gamma g(\exp(te))e,$$

that every $\Gamma \backslash (Ge)$, $e \in F_1$, is a closed subset of $B(M)$ invariant under T_t, and that the geodesic flow is ergodic if K acts transitively on E_1; this last result includes the classical theorem of E. Hopf on complete Riemann manifolds of constant negative curvature and of finite volume. Assume, now, that the total measure of M (or $B(M)$) be finite. It is then proved that the measure μ on $B(M)$ decomposes into measures concentrated on closed subsets of the form $\Gamma \backslash (Ge)$ and that, except for some special e's, the set $\Gamma \backslash (Ge)$ is ergodic under the geodesic flow with respect to the measure on $\Gamma \backslash (Ge)$ which is obtained from the above decomposition of μ and which can be also obtained from the Haar measure on $\Gamma \backslash G$ in a simple manner. Finally, the author also discusses the question whether the spectrum of the geodesic flow is absolutely continuous and gives a new proof to a result of Gelfand and Fomin [Uspehi Mat. Nauk (N.S.) 7 (1952), no. 1(47), 118–137; MR 14, 660; 17, 514]. *K. Iwasawa.*

Tonolo, Angelo. Sugli spazi riemanniani normali ad n dimensioni. Rend. Sem. Mat. Univ. Padova 26 (1956), 328–333.

The author takes up again a problem which he initiated [Pont. Acad. Sci. Acta 13 (1949), 29–53; Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6 (1949), 438–444; MR 11, 461]; namely the problem when the principal axes of a symmetric covariant tensor field of degree 2, with simple characteristic roots, in a three-dimensional Riemannian space, span fields of planes that are integrable. The problem was successively generalized by Schouten [Colloque de Géométrie Différentielle, Louvain, 1951, Thone, Liège, 1951, pp. 67–70; MR 13, 281] to n-dimensional spaces; by Nijenhuis [Nederl. Akad. Wetensch. Proc. Ser. A. 54 (1951), 200–212; MR 13, 281] to a field of linear transformations (with simple eigenvalues), without metric or connection; and after that, Haantjes [ibid. 58 (1955), 158–162; MR 16, 1151] weakened the simplicity of eigenvalues to diagonalizability of the automorphism; and he gave a very elegant form for the necessary and sufficient conditions for integrability of the planes spanned by the eigenvectors of one or more eigenvalues. Finally, Frölicher and Nijenhuis [ibid. 59 (1956), 338–359; MR 18, 569] gave a greatly simplified deduction of Haantjes' conditions.

The present author returns to the Riemannian case, in an n-dimensional space, takes the Ricci tensor for the covariant tensor field, and finds integrability conditions which are not of the form of the vanishing of one or more tensors (like Schouten's or Haantjes' conditions), but

requires the divisibility of a set of polynomials by the characteristic equation of the Ricci tensor. The eigenvalues are again assumed to be simple.

A. Nijenhuis (Seattle, Wash.).

Kostant, Bertram. Holonomy and the Lie algebra of infinitesimal motions of a Riemannian manifold. Trans. Amer. Math. Soc. **80** (1955), 528–542.

Let M be a Riemannian manifold of class C^∞ and V_0 the tangent space at a point o of M. We denote by X, Y, $\cdots$ vector fields on M and by x, y, $\cdots$ the respective values of these vector fields at o. We associate to X a field of tangent space endomorphisms A_X in the following way. For $v \in V_0$ we define the value a_X of A_X at o by $a_Xv=-\nabla_vX$, where ∇_v is the operator of covariant differentiation with respect to Christoffel symbols in the direction given by v. Let $\mathfrak{a}_0$ be the Lie algebra of all skew-symmetric endomorphisms of V_0 and $\hat{\mathfrak{g}}_0=\mathfrak{a}_0+V_0$. We introduce into $\hat{\mathfrak{g}}_0$ a bracket operation where

$$[a_1, a_2]=a_1a_2-a_2a_1,\ [a, v]=a(v),$$
$$[v, a]=-a(v),\ [v_1, v_2]=t(v_2, v_1)$$

for $a, a_1, a_2 \in \mathfrak{a}_0$ and $v, v_1, v_2 \in V_0$, where $t(v_2, v_1)$ is the contraction of the curvature tensor t at o on the last two indices by v_2 and v_1.

The author proves first the following theorem. Let $o \in M$. Let $\mathfrak{g}_0=\mathfrak{a}_0+V_0$ be the algebra defined in the above mentioned way. Let $\mathfrak{g}$ be the Lie algebra of infinitesimal motions on M. Let θ_0: $\mathfrak{g}\to\hat{\mathfrak{g}}_0$ be the mapping defined by $\theta_0(X)=X_0=a_X+x$ and $\mathfrak{g}_0=\theta_0(\mathfrak{g})$, then θ_0 is an isomorphism of $\mathfrak{g}$ onto $\mathfrak{g}_0$. This theorem generalizes the theorem given by E. Cartan where M is a symmetric Riemannian space.

Let σ_0 be the restricted homogeneous holonomy group and $\mathfrak{s}_0$ the Lie algebra of σ_0. $\mathfrak{s}_0$ is a subalgebra of $\mathfrak{a}_0$. We define the inner product in $\mathfrak{a}_0$ by $B(a_1, a_2)=\text{trace } a_1a_2$, then B is negative definite. We denote by u_0 the orthocomplement of $\mathfrak{s}_0$ in $\mathfrak{a}_0$ with respect to B. For any $X \in \mathfrak{g}$ we consider the decomposition of a_X, $a_X=b_X+e_X$, where $b_X \in \mathfrak{s}_0$ and $e_X \in u_0$. Let B_X (resp. E_X) be the field of tangent space endomorphisms which at any point o takes the value b_X (resp. e_X).

The author then proves the following lemma. The field E_X is covariant constant. Combining this lemma and the theorem of Green ingeniously, he proves next the following theorem. Let M be a compact Riemannian manifold and X an infinitesimal motion on M. Then $a_X \in \mathfrak{s}_0$. Lichnerowicz [Géométrie différentielles, Colloq. Internat. Centre Nat. Rech. Sci., Strasbourg, 1953, pp. 171–184; MR 16, 519] proved that, when $x=0$, the same result holds if $\mathfrak{s}_0$ acts irreducibly (that is, if M is irreducible) and the Ricci tensor is not zero.

Let $\mathfrak{g}^*$ be a subalgebra of $\mathfrak{g}$. Let $V(\mathfrak{g}_0^*)$ be the subspace of V_0 defined by

$$V(\mathfrak{g}_0^*)=\{x \in V_0 | X \in \mathfrak{g}^*\}.$$

We say that $\mathfrak{g}^*$ is transitive on M if $V(\mathfrak{g}_0^*)=V_0$ for every $o \in M$.

The author first proves the following lemma. Let S be a tensor field in M which is invariant under motions, i.e. $L_XS=0$ for all $X \in \mathfrak{g}$, where L_X is the symbol of Lie derivation with respect to X. Let W_0 be any subspace of the space of the mixed tensor algebra at $o \in M$, which is invariant under a_X for all $X \in \mathfrak{g}$ and such that $s \in W_0$. Then for any $X_1, X_2, \cdots, X_k \in \mathfrak{g}$ the value of the field $\nabla_{X_1}\nabla_{X_2}\cdots\nabla_{X_k}S$ at $o \in M$ is contained in W_0. Then he proves the following theorem. Assume $\mathfrak{g}^*$ is a Lie algebra of infinitesimal motions on M which is transitive. For any point $o \in M$, let $\mathfrak{h}_0^*$ be the Lie algebra of skew-symmetric endomorphisms of V_0 generated by all a_X for $X \in \mathfrak{g}^*$. Then $\mathfrak{s}_0 \subseteq \mathfrak{h}_0^* \subseteq n(\mathfrak{s}_0)$, where $n(\mathfrak{s}_0)$ is the normalizer of $\mathfrak{s}_0$.

Since we have already seen that, if M is compact or if M is irreducible and the Ricci tensor does not vanish, then $a_X \in \mathfrak{s}_0$ for all $X \in \mathfrak{g}$, combining these facts with the above theorem, we obtain the following theorem. Let $\mathfrak{g}^*$ be any Lie algebra of infinitesimal motions which is transitive on M. If either one of the following conditions hold: (a) M is compact, or (b) M is irreducible and the Ricci tensor does not vanish, then we have $\mathfrak{h}_0^*=\mathfrak{s}_0$.

K. Yano (Tokyo).

Solodovnikov, A. S. Projective transformations of Riemannian spaces. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 4(70), 45–116. (Russian)

A detailed exposition, with proofs, is given of results announced in part in Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 419–422 [MR **17**, 783]. A vector field ξ^i in a Riemannian manifold defines an infinitesimal one-parameter group of projective transformations (preserving the point sets of geodesics) if and only if there exists a scalar φ such that

$$\xi_{(i,j)k}=2g_{ij}\varphi_{,k}+g_{ik}\varphi_{,j}+g_{jk}\varphi_{,i}.$$

If $\xi_{(i,j)}\neq\rho g_{ij}$ (otherwise the map is affine), there is a coordinate system in which the line element is of the form

$$ds^2=\sum_{\alpha=1}^{p}\prod_\beta{}' |f_\beta-f_\alpha| ds_\alpha{}^2=\sum_{\alpha=1}^{p}\Phi_\alpha,$$
$$\xi_{(i,j)}dx^idx^j=\sum_{\alpha=1}^{p}\varphi_\alpha\Phi_\alpha,\ \varphi_\alpha=f_\alpha+\sum_1^p f_\beta.$$

($\prod'_\beta$ is the product over $\beta\neq\alpha$.) This generalization of the Liouville form is called a Levi-Cività metric [Levi-Cività, Ann. Mat. Pura Appl. (2) **24** (1896), 255–300]. The coordinates $x^1, \cdots, x^n$ split into p ($p>1$) groups, (x^{i_α}), such that $ds_\alpha{}^2$ is a positive definite form in the x^{i_α}'s alone; f_α is a function only of x^{i_α}, unless the group belonging to α contains more than one coordinate, in which case f_α is constant. The space is called of basic type if the associated p-dimensional form

$$ds^2=\sum_{\alpha=1}^{p}\prod_\beta{}' |f_\beta-f_\alpha|(dy_\alpha)^2$$

does not have constant curvature, otherwise it is called exceptional. Equations are given for determining all metrics of basic type; curiously, no more than two f_α's may be constant.

For the exceptional metrics, the constant curvature K does not depend on the particular decomposition into the required form and each $ds_\alpha{}^2$ has constant curvature. An essentially unique maximal K-decomposition is defined which facilitates the description of all exceptional metrics and their groups.

Finally affine transformations are allowed, and a description of all metrics admitting a projective group larger than the group of motions is given. The results include, among others, those of de Vries [Math. Z. **60** (1954), 328–347; MR **16**, 168] and the authors he cites.

L. W. Green (Minneapolis, Minn.).

Pan, T. K. Indicatric torsion in a subspace of a Riemannian space. Proc. Amer. Math. Soc. **8** (1967), 294–298.

The author has already [same Proc. **7** (1956), 449–457; MR **18**, 64] defined a notion called "indicatric torsion of a

vector field" which generalizes the geodesic torsion of a curve in ordinary space.

In this note he considers a subspace V_m of a Riemannian space V_m, and in this subspace a unit vector field v and a curve C. Regarding v and C as in V_m, the author writes down the formulae of Frenet for v along C in V_m, calling ${}_vK_p$ the associate curvature of order p of the vector field v for the curve C. He calls ${}_vK_2$ the generalization of the indicatric torsion which he had previously introduced. Various expressions are given for this scalar and theorems proved for a few special cases. *E. T. Davies.*

Sumitomo, Takeshi. On some transformations of Riemannian spaces. Tensor (N.S.) 6 (1956), 136–140.

Consider a Riemannian space M with positive definite fundamental tensor g_{ji}. An infinitesimal transformation v^h is said to be conformal if it satisfies

$$\underset{v}{\mathscr{L}} g_{ji} = \nabla_j v_i + \nabla_i v_j = 2\varphi g_{ji},$$

where $\underset{v}{\mathscr{L}}$ denotes the Lie derivative with respect to v^h, ∇_j the covariant differentiation and φ a scalar. When φ is a constant, the conformal transformation is said to be homothetic. A homothetic transformation is characterized by $\underset{v}{\mathscr{L}} g_{ji} = 2\varphi g_{ji}$ and $\underset{v}{\mathscr{L}} \left\{ {h \atop ji} \right\} = 0$, where $\left\{ {h \atop ji} \right\}$ are Christoffel symbols. It is well-known that in a compact orientable M, a homothetic transformation is a motion.

The author calls an infinitesimal transformation pseudo-homothetic if it satisfies $\underset{v}{\mathscr{L}} g_{ji} = 2\varphi g_{ji}$ and $\underset{v}{\mathscr{L}} K_{kji}^{\cdots h} = 0$, where $K_{kji}^{\cdots h}$ is the curvature tensor formed with $\left\{ {h \atop ji} \right\}$.

He proves: (1) In a compact M, pseudo-homothetic transformations are homothetic. (2) In a compact M with zero scalar curvature, conformal transformations are homothetic. (3) In a compact M with negative constant scalar curvature, conformal transformations are motions. (4) In an M with $|K_j^h| \neq 0$, pseudo-homothetic transformations are homothetic. (5) In an M with scalar curvature different from zero, a necessary and sufficient condition that a conformal transformation be homothetic is that it leave $K_{kji}^{\cdots h}$ and $\nabla_1 K_{kji}^{\cdots h}$ invariant. He studies then the two-dimensional cases.

In the last two sections, the author proves the following three theorems. (6) A necessary and sufficient condition that the vector v^h be concurrent is that the infinitesimal transformation v^h be pseudo-homothetic and its trajectories be geodesics. (7) In a compact connected M, a projective collineation preserving curvature tensor is affine. (8) In a compact simply connected M with zero Ricci curvature, a projective collineation is a motion. *K. Yano* (Tokyo).

Kawaguchi, Akitsugu. On the theory of non-linear connections. II. Theory of Minkowski spaces and of non-linear connections in a Finsler space. Tensor (N.S.) 6 (1956), 165–199.

The author applies his earlier work on non-linear connections [Tensor (N.S.) 2 (1952), 123–142; MR 14, 585] to the study of a Finsler space. There have been two methods of approach to the problem of establishing a theory of connection in Finsler spaces. One of them is represented in the work of E. Cartan [Les espaces de Finsler, Hermann, Paris, 1934] in which the space is regarded as made up of line-elements and the connection is linear in the increments of the point coordinates as well as in the increments of the components of the line elements attached to each point. The space is regarded as locally Euclidean in the neighborhood of each line-element. The other point of view is represented in recent papers by Rund [Math. Z. 54 (1951), 115–128; MR 13, 159], Wagner [Trudy Sem. Vektor. Tenzor. Anal. 7 (1949), 65–166; MR 13, 717] and Barthel [Convegno Internazionale di Geometria Differenziale, Italia, 1953, Edizioni Cremonese, Roma, 1954, pp. 71–76; MR 16, 173]. It amounts to regarding the Finsler space as locally Minkowskian. The corresponding connection is then non-linear.

The present paper is a contribution to the locally Minkowskian approach. The first few paragraphs deal with Minkowski spaces, and the author defines "relative" quantities and Euclidean spaces, and points out that the Cartan approach to Finsler spaces may be described as the theory of a $(2n-1)$-parameter family of relative Euclidean spaces. A fairly detailed study of the indicatrix is made using the notions of affine geometry. Conditions are given in order that Minkowski space shall be Euclidean.

The second part of the paper starts off with the definition of tangent Minkowski spaces to a Finsler space. The conditions in order that the Finsler space may be Riemannian are those which ensure that the tangent Minkowski spaces shall be Euclidean. Connection in the Finsler space is defined as a correspondence between the tangent Minkowski spaces at two infinitely near points, which arises naturally from his definition of infinitesimal motions in a Minkowski space. This leads to a three-index set of connection functions depending both on the coordinate x^i of a point, and on the component X^i of a vector. The functions are homogeneous of degree zero in X^i. They, in turn, lead to notions of absolute differential and covariant derivation. The author gives some relations between his theory and that of other authors.
E. T. Davies (Southampton).

Nomizu, Katsumi. On infinitesimal holonomy and isotropy groups. Nagoya Math. J. 11 (1957), 111–114.

The infinitesimal isotropy group K_p is the group of linear transformations of the tangent space $T_p(M)$ which leave invariant the tensors T, ∇T, $\nabla^2 T$, $\cdots$, R, ∇R, $\nabla^2 R$, $\cdots$, at p. Just as the isotropy group carries the holonomy groups into themselves, so K_p carries the infinitesimal holonomy group H_p' into itself. The main result of the paper is: $\nabla T = \nabla R = 0$ if (1) $H_p' \subset K_p$ at all p in M, and (2) H_p' is irreducible at every p of M. A corollary states that if G/H is the homogeneous space of a Lie group, which has an invariant affine connection, then $\nabla T = \nabla R = 0$ if H_p' is irreducible and is contained in the linear isotropy group determined by H. The latter is a correction of a statement in a previous paper [Nagoya Math. J. 9 (1955), 57–66; MR 17, 891]. *A. Nijenhuis.*

Couty, Raymond. Sur les transformations définies par le groupe d'holonomie infinitésimale. C. R. Acad. Sci. Paris 244 (1957), 553–555.

The Lie algebra of the infinitesimal holonomy group H_p' [Nijenhuis, Nederl. Akad. Wetensch. Proc. Ser. A. 56 (1953), 233–240, 241–249, 57 (1954), 17–25; MR 16, 171, 172] at $p \in M$ is spanned by the tensors

$$(\overset{0}{\Omega}{}_k{}^i)_0 = (R_k{}^i{}_{mn} U^m V^n)_0, \cdots,$$

$$(\overset{p}{\Omega}{}_k{}^i)_0 = (\nabla_{a_1} \cdots \nabla_{a_p} R_k{}^i{}_{mn} U^m V^n W_1{}^{a_1} \cdots W_p{}^{a_p})_0, \cdots;$$

and $\overset{0}{\mathscr{T}}, \cdots, \overset{p}{\mathscr{T}}, \cdots$ denote the corresponding transformations of a neighborhood U of p by having H_p' act on the normal coordinates of the points of U. An $\mathscr{H}$ space satisfies $H_{ijkl;mn}=(\nabla_m\nabla_n-\nabla_n\nabla_m)R_{ijkl}$. Results are: (1) If the elements $\overset{0}{\mathscr{T}}$ are affine, M is symmetric, M is an $\mathscr{H}$ space; if the Ricci curvature is non-degenerate and $\overset{0}{\mathscr{T}}$, $\overset{1}{\mathscr{T}}$ are affine, M is symmetric; if M is positive definite Riemannian, and $\overset{1}{\mathscr{T}}, \overset{2}{\mathscr{T}}$ are affine, M is symmetric. (2) If M is an Einstein space and $\overset{0}{\mathscr{T}}$ conformal then M is an $\mathscr{H}$ space; if M is an Einstein space, M positive definite Riemannian and $\overset{1}{\mathscr{T}}, \overset{2}{\mathscr{T}}$ conformal, then M is symmetric.

A. Nijenhuis (Seattle, Wash.).

Chaki, M. C. On the line geometry of a curvature tensor. Bull. Calcutta Math. Soc. **47** (1955), 217–226.

L'A. se propose ici d'étendre au tenseur de courbure d'une connexion affine les résultats de Ruse [Proc. Roy. Soc. Edinburgh. Sect. A. **62** (1943–44), 64–73, MR **6**, 106] relatifs au tenseur de courbure d'une connexion riemannienne. Sur une variété différentiable V_n sont donnés une métrique riemannienne g_{ij} et une connexion affine Γ^i_{jk}; une direction X^i $(i=1, \cdots, n)$ tangente en x à V_n définit un point de l'espace projectif S_{n-1} à l'infini de l'espace tangent en x, le tenseur fondamental g_{ij} définit une $(n-2)$-quadrique non dégénérée de S_{n-1} d'équation ponctuelle $g_{ij}X^iX^j=0$ et d'équation tangentielle $g^{ij}u_iu_j=0$. S_tX^i, Y^i sont deux points de S_{n-1}, $p^{ij}=X^iY^j-X^jY^i$ sont les coordonnées de la droite joignant ces deux points; les droites de S_{n-1} satisfaisant à $F_{ij,kl}p^{ij}p^{kl}=0$ ($F_{ij,kl}$ tenseur de courbure de la connexion Γ rendu covariant à l'aide de la métrique) définissent un complexe quadratique de S_{n-1}. L'A. étudie les propriétés géométriques de ce complexe relativement à la quadrique fondamentale, principalement dans les cas $n=3$ et $n=4$. Les particularités obtenues pour une connexion self-associée ou self-conjuguée au sens de Sen [Bull. Calcutta Math. Soc. **42** (1950), 177–187; MR **12**, 533] sont signalées. On obtient ainsi une extension des principaux résultats de Ruse.

A. Lichnerowicz (Paris).

Fujimoto, Atsuo. On decomposable symmetric affine spaces. J. Math. Soc. Japan **9** (1957), 158–170.

An n-dimensional affinely connected space A_n without torsion is symmetric in the Cartan sense if the reflection about any point in A_n is an affine collineation. Such a space A_n always admits a transitive group G_{n+r} of affine collineations consisting of a set of transvections and an isotropic subgroup. Specifically, if the $(n+r)$-parameter continuous transformation group G_{n+r} satisfies certain structural equations, an involutive automorphism σ of G_{n+r} and an isotropic subgroup H_r invariant under σ can be defined. The group G_{n+r} is said to be effective if H_r does not contain any invariant subgroup of G_{n+r}. Let coordinates on A_n be denoted by $x^1, x^2, \cdots, x^n$; let A_p and A_{n-p} denote the spaces on which the coordinates are $x^1, x^2, \cdots, x^p$ and $x^{p+1}, x^{p+2}, \cdots, x^n$, respectively. The indices of the coordinates of A_p and A_{n-p} are indices of the first and second kinds, respectively. A_n is called the product space of A_p and A_{n-p}, and is said to be decomposable if those components of the connection $\Gamma_{ij}{}^k$ of A_n vanish whose indices i, j, k are not all of the same kind.

The following theorems are proved. Theorem 1. If an effective group G_{n+r} is given, there always exists an n-dimensional affine space A_n whose complete group of affine collineations contains the subgroup isomorphic to G_{n+r}. Theorem 2. A decomposable affinely connected space without torsion is symmetric if each composition space is symmetric. Theorem 3. A symmetric affine space A_n is decomposable if and only if there exists a coordinate system such that the components of the curvature tensor evaluated at any point in this coordinate system are breakable (i.e. those components of the curvature tensor vanish whose indices are of two kinds). Theorem 4. The symmetric A_n determined by an effective group G_{n+r} is a product space of A_p and A_{n-p} of dimensions p and $n-p$, respectively, if and only if the n-dimensional vector space V spanned by the transvections of G_{n+r} is a direct sum of a p-dimensional subspace V_1 and an $(n-p)$-dimensional subspace V_2 satisfying the following conditions: (1) $[X_{i^1}, X_{i^2}]=0$; (2) $[[X_{i^\lambda}, X_{j^\lambda}], X_{k^\lambda}]$ $(\lambda=1, 2)$ are linear combinations of X_{l^λ} only, where X_{i^λ} are the base of V_λ $(\lambda=1, 2)$ and $i^1, j^1, k^1, l^1=1, 2, \cdots, p$, and $i^2, j^2, k^2, l^2=p+1, \cdots, n$. Theorem 5. If the symmetric space A_n determined by G_{n+r} is decomposable, where G_{n+r} is semi-simple and effective, then G_{n+r} is decomposed into a direct product of two groups by which composition spaces are determined. Corollary 1. If G_{n+r} is simple, semi-simple, and effective, then the symmetric space A_n determined by G_{n+r} is non-decomposable. Corollary 2. If G_{n+r} is effective and the linear adjoint group $\bar{H}_r$ (corresponding to H_r and acting on the transvections of G_{n+r}) is irreducible, then the symmetric space A_n determined by G_{n+r} is either flat or non-decomposable.

P. O. Bell.

Izmaĭlov, V. D. The problem of the interior normalization of a hypersurface in a space of affine connectivity. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 906–909. (Russian)

Let A_n be a space of affine connection, with object of affine connection Γ^i_{jk}, let G^i_{jk} be the symmetric part of Γ^i_{jk}, and let X_{n-1}: $\xi^i=\xi^i(u^1, \cdots, u^{n-1})$ $(i=1, \cdots, n)$ be a hypersurface in A_n. Let $B^i_\alpha=\partial\xi^i/\partial u^\alpha$ and $B^i_{\alpha\beta}=\partial_\beta\partial_\alpha\xi^i$, let the vector m_i be a non-null solution of $B_\alpha{}^i m_i=0$, and consider the symmetric asymptotic tensor

$$\omega_{\alpha\beta}=(B^i_{\alpha\beta}+G^i_{jk}B^j_\alpha B^k_\beta)m_i.$$

The hypersurface X_{n-1} is said to be non-degenerate provided that det $|\omega_{\alpha\beta}|\neq 0$. The author establishes, for non-degenerate $X_2 \subset A_3$, the existence of an invariant vector field over the surface that determines an interior connection on the surface, and then treats the rather more complicated general case $n\geqq 3$. Special cases reduce to known results; see, for instance, the paper by A. M. Lopšic [Trudy Sem. Vektor. Tenzor. Anal. **8** (1950), 273–285; MR **12**, 636].

E. F. Beckenbach.

Mutō, Yosio. On n-dimensional projectively flat spaces admitting a group of affine motions of order $r=n^2-n+1$. Sci. Rep. Yokohama Nat. Univ. Sect. I. **5** (1956), 1–15.

Using the method of Lie derivatives, the author determined in a previous paper [same Rep. **4** (1955), 1–18; MR **17**, 783] all n-dimensional projectively flat manifolds A_n with symmetric affine connexion, and which admit a complete group of affine motions G_r of order $r>n^2-n$. In the paper under review, the author studies the case $r=n^2-n+1$ and proves the following theorems.

A necessary and sufficient condition that a projectively flat manifold A_n $(n\geqq 3)$ with symmetric affine connexion, and with (I) asymmetric Ricci tensor, (II) negative semi-

definite symmetric Ricci tensor, (III) positive semi-definite Ricci tensor, and (IV) indefinite symmetric Ricci tensor, admits a complete group of affine motions of order $r=n^2-n+1$ is that the connexion parameters satisfy respectively

$$\text{(I)}\quad \Gamma^{\lambda}_{\mu\nu}=-\delta^{\lambda}_{\mu}\Psi_{\nu}-\delta^{\lambda}_{\nu}\Psi_{\mu},\ \Psi_1=a\gamma x^2,\ \Psi_2=-a\gamma x^1,\ \Psi_3=\cdots=\Psi_n=0\ (a\gamma^2=-1;\ a=\pm 1),$$

$$\text{(II)}\quad \Gamma^{\lambda}_{\mu\nu}=-\delta^{\lambda}_{\mu}\Psi_{,\nu}-\delta^{\lambda}_{\nu}\Psi_{,\mu},\ \Psi=\tfrac{1}{2}\log(x^1x^2-1),$$

$$\text{(III)}\quad \Gamma^{\lambda}_{\mu\nu}=-\delta^{\lambda}_{\mu}\Psi_{,\nu}-\delta^{\lambda}_{\nu}\Psi_{,\mu},\ \Psi=\tfrac{1}{2}\log[1+(x^1)^2+(x^2)^2],$$

$$\text{(IV)}\quad \Gamma^{\lambda}_{\mu\nu}=-\delta^{\lambda}_{\mu}\Psi_{,\nu}-\delta^{\lambda}_{\nu}\Psi_{,\mu},\ \Psi=\tfrac{1}{2}\log(1-x^1x^2)$$

in some coordinate system.

Combining the results obtained previously with those obtained in the present paper, the author states the following theorem. Consider a manifold A_n ($n\geqq 5$) or a projectively flat manifold A_n ($n\geqq 3$) with symmetric affine connexion. A necessary and sufficient condition that the manifold admits a complete group of affine motions of order $r=n^2-n+1$ is that the manifold be one of four spaces described in the above theorem.

K. Yano (Tokyo).

See also: Laugwitz, p. 927.

Complex Manifolds

Stein, Karl. Überlagerungen holomorph-vollständiger komplexer Räume. Arch. Math. **7** (1956), 354–361.

Ein komplexer Raum heisst holomorph vollständig, wenn er holomorph-separabel und zugleich holomorph-konvex ist [siehe H. Grauert, Math. Ann. **129** (1955), 233–259; MR **17**, 80]. Die Bedingung "holomorph-vollständig" ist für komplexe Räume besonders wichtig. So gelten für holomorph-vollständige komplexe Räume noch die Approximationssätze und die Cartansche Theorie der kohärenten analytischen Garben. Nach einem grundlegenden Satz von K. Oka ist auch jedes lokal-pseudokonvexe unverzweigte Riemannsche Gebiet über dem C^n vollständig.

Verf. beweist nun in der folgenden Arbeit, dass jede unverzweigte und unbegrenzte Überlagerung eines holomorph-vollständigen komplexen Raumes wieder holomorph vollständig ist. Damit ist auch eine Frage für komplexe Mannigfaltigkeiten von J.-P. Serre [Colloque sur les fonctions de plusieurs variables, Bruxelles, 1953, Thone, Liège, 1953, pp. 57–68; MR **16**, 235] positiv beantwortet.

H. Behnke (Münster).

Burdina, V. I. Real characteristic cycles of complex manifolds. Mat. Sb. N.S. **39(81)** (1956), 337–378. (Russian)

This paper brings the detailed proofs of results announced earlier [Dokl. Akad. Nauk SSSR (N.S.) **96** (1954), 1085–1088; MR **16**, 736]. The intersection ring of the Grassmann manifold $H(2k, 2l)$ of $2k$-planes in $2(k+l)$-space in dimensions $\geqq 4kl-2k$ is described, using Pontryagin's description of the cycles by means of non-decreasing functions from $\{1, \cdots, 2k\}$ to $\{0, 1, \cdots, 2l\}$. Integers and integers mod 2 are used as coefficients. Similar results are developed for the complex Grassmann manifold $C(k, l)$. Finally the intersections of the cycles of $H(2k, 2l)$ with the submanifold $C(k, l)$ are studied. These intersections enter into the determination of the characteristic cycles (Pontryagin cycles) of complex manifolds.

H. Samelson (Ann Arbor, Mich.).

Bernard, Daniel. Sur les G-structures complexes. C. R. Acad. Sci. Paris **243** (1956), 1821–1824.

A G-structure over a real or complex manifold (differentiable, real, or complex analytic) is, in the language of fiber bundles, a reduction of the structural group of the tangent bundle to a subgroup G of the linear group. A real G-structure determines a complex G-structure. Conditions are established that a complex G-structure be equivalent to a real one; for instance, this is the case when G is compact. Let K denote the real or complex field, K^n be the n-dimensional vector space over K and $(K^n)^*$ its dual space. Then $P=K^n\otimes\overset{2}{\Lambda}(K^n)^*$ is acted on by G, the linear map corresponding to $g\in G$ being $g^{-1}\otimes\overset{2}{\Lambda}g$. The author introduces a tensor, with values in a quotient space of P, and defined in the principal G-bundle space. For an almost complex structure the vanishing of this tensor gives the integrability conditions.

S. Chern.

Gunning, R. C. The structure of factors of automorphy. Amer. J. Math. **78** (1956), 357–382.

Le présent article développe des résultats annoncés dans une note précédente [Proc. Nat. Acad. Sci. U.S.A. **44** (1955), 496–498; MR **17**, 84]. D est une variété analytique complexe simplement connexe, à n dimensions complexes, ayant un groupe dénombrable Γ d'automorphismes analytiques sur lequel on suppose: chaque point $z\in D$ possède un voisinage U tel que $TU=U$ ou $TU\cap U=\emptyset$ pour toute transformation $T\in\Gamma$; si z_1 et z_2 appartiennent à D, ou bien $\Gamma z_1=\Gamma z_2$, ou bien il existe des voisinages U_1, U_2 de ces points tels que $\Gamma U_1\cap\Gamma U_2=\emptyset$; l'espace quotient D/Γ est compact; il existe une métrique kaehlérienne sur D invariante par Γ. On considère alors les fonctions holomorphes dans D, $\sigma_T(z)$ associées aux $T\in\Gamma$, et telles que

$$\sigma_{ST}(z)=\sigma_S(Tz)+\sigma_T(z)$$

et les fonctions $\nu_T(z)$ holomorphes et ne s'annulant pas sur D telles que

$$\nu_{ST}(z)=\nu_S(Tz)\cdot\nu_T(z).$$

Ces derniers sont les facteurs d'automorphie. Un facteur μ détermine une classe χ_1 de caractères. Une condition nécessaire et suffisante pour que μ et ν soient associées tous deux à χ_1 est l'existence d'un caractère (c_T) du groupe Γ et celle d'une fonction σ avec $\mu_T(z)=c_T\nu_T(z)\exp\sigma_T(z)$; pour que deux facteurs soient équivalents, il faut et il suffit qu'ils aient la même classe de caractères. Une fonction $f(z)$ méromorphe sur D est dite automorphe relativement au facteur ν si $f(Tz)=\nu_T(z)f(z)$. L'auteur utilise une représentation de Γ comme groupe isomorphe à H/P, où H est un groupe libre, modulo un groupe de relations P (à nombre fini de générateurs) et l'homomorphisme

$$\varphi:\ P\cap[H, H]/[P, H]\rightarrow H_2(D/\Gamma)/S_2(D/\Gamma).$$

Pour tout facteur ν, f étant associée à ν, on a

$$c(f)[\varphi(\nu)]=\chi_1(\nu)[\nu]$$

pour tout élement $\nu\in P\cap[H, H]$. Un facteur ν est dit positif s'il existe une associée $f(z)$ holomorphe sur D. Si Γ ne renferme pas de transformation à points fixes, on peut associer à un tel facteur ν, et à $\chi_1(\nu)$, considéré comme un 2-cocycle de D/Γ, une matrice hermitienne positive. Si g est une fonction C^∞ dans D pour laquelle $d_{\bar z}d_z g$ est invariant par Γ, χ_g prenant des valeurs entières, les multiples de g constituent les classes de caractères des facteurs d'automorphie.

P. Lelong (Paris).

Hano, Jun-ichi; and Matsushima, Yozô. Some studies on Kaehlerian homogeneous spaces. Nagoya Math. J. 11 (1957), 77–92.

Le point de départ de cette étude est la décomposition de De Rham d'un espace kählérien complet simplement connexe, à métrique analytique, en facteurs kählériens, soit $V=V_0\times V_1\times\cdots\times V_n$. Si V est homogène, il en est de même de chacun des facteurs V_i. Dans le cas de l'espace homogène kählérien G/B d'un groupe de Lie réductif G, cette décomposition s'identifie à celle établie par A. Lichnerowicz [Géométrie différentielle, Colloq. Internat. Centre Nat. Rech. Sci., Strasbourg, 1953, pp. 171–184; C. R. Acad. Sci. Paris **242** (1956), 1410–1413; MR **16**, 519; **18**, 508] et A. Borel [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 1147–1151; MR **17**, 1108].

Si G est semi-simple, il est possible de définir une connexion affine canonique de 1ère espèce au sens de K. Nomizu [Amer. J. Math. **76** (1954), 33–65; MR **15**, 468]; et si cette connexion coïncide avec celle qu'induit la métrique kählérienne, alors G/B est hermitien symétrique. Extension au cas où G est seulement supposé réductif.

J. Lelong (Paris).

Dalla Volta, V. Varietà totalmente geodetiche nello spazio delle matrici simmetriche. Rend. Mat. e Appl. (5) 13 (1955), 294–334.

In the Siegel-Hua half-space of real dimension $p(p+1)$, with any geodesic curve emanating from a fixed point there are associable p real numbers λ_s $(\lambda_1^2+\cdots+\lambda_p^2=1)$, which are the elements of a certain reduced diagonal matrix corresponding to it. First, the author proves that there exists a totally geodesic holomorphic two-dimensional surface V_2 tangent to a geodesic if and only if for some $q<p$, $\lambda_1=\cdots=\lambda_q=0$, $\lambda_{q+1}=\cdots=\lambda_p$. Each such surface has constant negative curvature (as, more generally, in all symmetric Kähler spaces) which has the value $-(p-q)^{-1}$. If two V_2 have the same curvature they are symplectically homeomorphic.

Higher dimensional totally geodesic holomorphic subspaces V_{2r} are much harder to deal with since their tangency to a geodesic depends on the multiplicities of the $\{\lambda_s\}$ occurring, and even when the existence of a V_{2r} tangent to a geodesic can be asserted the uniqueness still remains in question. Relatively easy is the case $p=2$, that is $p(p+1)=6$. In this case there exist infinitely many V_4, and any two are symplectically homeomorphic. There is one and only one V_4 tangent to a given generic geodesic, but "generic" means that certain exceptional directions must be left out of consideration, even now.

Finally, the author makes some statements on totally geodesic subspaces which are not necessarily holomorphic, and at one stage of the discussion he uses the fact that any complex homeomorphism of the total space is of necessity linear and symplectic.

S. Bochner.

Satake, Ichiro. On the compactification of the Siegel space. J. Indian Math. Soc. (N.S.) 20 (1956), 259–281.

Soit $\mathscr{H}_n$ le demi-plan généralisé de Siegel, formé des matrices symétriques Z à n lignes et n colonnes, à termes complexes, telles que $\operatorname{Im} Z$ soit définie positive. Le groupe modulaire M_n opère dans $\mathscr{H}_n$; l'espace-quotient $\mathscr{V}_n=M_n\backslash\mathscr{H}_n$ est un espace analytique, et même une V-variété au sens de Satake. Comme bien connu, $\mathscr{V}_n$ n'est pas compact; il est classique que $\mathscr{V}_1$ se compactifie par adjonction d'un point à l'infini $\mathscr{V}_0$; récemment, Satake [Proc. Internat. Symposium Algebraic Number Theory, Tokyo and Nikko, 1955, Science Council of Japan, Tokyo, 1956, pp. 107–129; MR **18**, 731] a défini une compactification $\bar{\mathscr{V}}_2$ de $\mathscr{V}_2$ en utilisant la théorie de Minkowski pour la réduction des formes quadratiques; $\bar{\mathscr{V}}_2$ est une V-variété réalisable comme sous-variété algébrique dans un espace projectif (Baily). Dans le présent travail, Satake définit une compactification $\mathscr{V}_n^*$ de $\mathscr{V}_n$ pour tout n; pour $n=2$, $\mathscr{V}_2^*$ est un quotient de $\bar{\mathscr{V}}_2$; dans le cas général, $\mathscr{V}_{n-1}^*$ s'identifie à un sous-espace fermé de $\mathscr{V}_n^*$, dont le complémentaire est $\mathscr{V}_n$, et on a donc $\mathscr{V}_n^*=\mathscr{V}_n\cup\mathscr{V}_{n-1}\cup\cdots\cup\mathscr{V}_0$. Pour définir la topologie de $\mathscr{V}_n^*$, l'auteur utilise le domaine fondamental de M_n et la théorie de Siegel.

Pour tout entier pair $m\geq 0$, on a la notion de forme modulaire de poids m; d'une façon précise, pour tout ouvert U de $\mathscr{V}_n$, on a un espace $\mathscr{A}_m^{(n)}(U)$ de formes modulaires sur l'ouvert de $\mathscr{H}_n$, image réciproque de U; les $\mathscr{A}_m^{(n)}(U)$ définissent un faisceau $\mathscr{A}_m^{(n)}$ sur $\mathscr{V}_n$. Tout $f\in\mathscr{A}_m^{(n)}(U^*\cap\mathscr{V}_n)$, U^* un ouvert de $\mathscr{V}_n^*$, définit, pour chaque entier $r<n$, un élément $\Phi_r^n(f)\in\mathscr{A}_m^{(r)}(U^*\cap\mathscr{V}_r)$. En fait, l'auteur considère (cf. coroll. 2 du th. 3) une topologie sur $\mathscr{H}_n^*=\mathscr{H}_n\cup\mathscr{H}_{n-1}\cup\cdots\cup\mathscr{H}_0$ (cette topologie est définie par la donnée de „filtres convergents"; il semble au rapporteur que la topologie de $\mathscr{V}_n^*$ n'est autre que la topologie-quotient de celle de $\mathscr{H}_n^*$). Si on note Φ_n^* l'application naturelle: $\mathscr{H}_n^*\to\mathscr{V}_n^*$, et si f est une forme modulaire holomorphe dans $\mathscr{H}_n\cap\Phi_n^{*-1}(U^*)$, f se prolonge par continuité à $\Phi_n^{*-1}(U^*)$; alors la restriction de ce prolongement à $\mathscr{H}_r\cap\Phi_n^{*-1}(U^*)$ n'est autre que $\Phi_r^n(f)$. L'auteur définit ainsi, sur l'espace $\mathscr{V}_n^*$, le faisceau $\mathscr{A}_m^{(n)*}$ des formes modulaires de poids m: pour chaque ouvert $U^*\subset\mathscr{V}_n^*$, on a un espace $\mathscr{A}_m^{(n)*}(U^*)$ formé de fonctions sur $\Phi_n^{*-1}(U^*)$. En particulier, le faisceau $\mathscr{A}_0^{(n)*}$ définit la structure d'espace analytique de $\mathscr{V}_n^*$. L'auteur affirme (sans démonstration) que $\mathscr{V}_n^*$ est bien un espace analytique, mais n'est pas une V-variété pour $n\geq 2$; il ignore si $\mathscr{V}_n^*$ est réalisable comme sous-variété algébrique d'un espace projectif.

H. Cartan.

Algebraic Geometry

Gibson, R. O.; and Semple, J. G. Cayley models of some homaloidal curve-systems of S_3. Proc. London Math. Soc. (3) 7 (1957), 75–86.

The authors construct the Cayley models of four homaloidal systems of curves in S_3 and identify these models. The cases considered are: (i) the system of conics through a fixed point which meet a fixed conic in two points; the Cayley image is the projective model of the linear system cut on a general quadric Ω of S_5 by quadrics through a generating place; (ii) twisted cubics through three fixed points and having a fixed chord; the Cayley image is the projective model of quadric sections of Ω through four points, three of which are joined to the fourth by lines on Ω; (iii) twisted cubics through four points; the Cayley image is the projective model of the sections of Ω by cubic primals containing four generating planes of the same system; (iv) twisted cubics with four fixed chords; the Cayley image is the projective model of sections of Ω by cubics with nodes at four fixed points of Ω.

J. A. Todd.

d'Orgeval, B. Surfaces elliptiques avec un faisceau elliptique de courbes de genre 4. II. Publ. Sci. Univ. Alger. Sér. A. 2 (1955), 205–240 (1957).

Continuation d'une recherche sur les surfaces elliptiques F possédant un faisceau elliptique de courbes C de genre

4 [cf. mêmes Publ. 1 (1954), 313–336; MR 17, 664]; comme les trajectoires du groupe des transformations birationnelles de F en elle-même découpent sur une courbe C une involution γ_n^1, il faut chercher avant tout les courbes de genre 4 qui possèdent une telle γ_n^1 (qui est engendrée par un groupe de transformations cyclique ou abélien de base 2). On représente C sur la courbe canonique Γ d'ordre 6, intersection d'une quadrique Q et d'une surface du troisième ordre; le cas où C serait hyperelliptique a déjà été traité; il faut alors trouver toutes les courbes canoniques Γ qui sont transformées en elles-mêmes par des homographies. On trouve que ces homographies doivent avoir au moins un point uni M sur Q; en projectant Q de M sur un plan on arrive à une question de géométrie dans le plan, c'est-à-dire: trouver les courbes planes du 6me ordre avec deux points triples ou bien du 5me ordre avec deux points doubles qui sont transformées en elles-mêmes par un groupe d'homographies, qui soit cyclique ou abélien de base 2. Après la détermination de toutes ces courbes, l'A. passe à discuter l'existence des correspondantes surfaces F; pour chacune de ces surfaces il trouve enfin la valeur de l'invariant S de Severi et le degré N de la représentation de F sur son image dans la variété de Picard. On peut remarquer qu'il y a des courbes C possédant un groupe abélien de base 2 de transformations birationnelles en elles-mêmes et qui ne peuvent appartenir à aucune surface F du type désiré.

E. Togliatti (Gênes).

Godeaux, Lucien. Remarques sur la formation des systèmes canonique et pluricanoniques de quelques surfaces algébriques. Acad. Roy. Belg. Bull. Cl. Sci. (5) 42 (1956), 1102–1106.

Teixidor, J. On curves which are the simple complete intersection of two algebraic surfaces. Abh. Math. Sem. Univ. Hamburg 20 (1956), 155–164. (Spanish)

Let F [Φ] be an algebraic surface of order m [n] in complex projective 3-space; $m\geqq n$. The skew irreducible curve Γ is assumed to be the simple complete intersection of F and Φ [cf. Severi, Abh. Math. Sem. Hansischen Univ. 15 (1943), 97–119; MR 7, 476]. Let G_{mn} denote the set of the $m\cdot n$ mutually distinct points P_j of intersection of Γ with a general plane π. The tangents of Γ at the P_j's are shown to satisfy a system of exactly $n(m-1)$ independent conditions. They are discussed by means of a method previously used by B. Segre [Ann. Mat. Pura Appl. (4) 26 (1947), 1–26; MR 10, 397].

Introduce affine coordinates x, y, z such that the infinite plane is in general position with respect to Γ and π has the equation $z=0$. Let $f(x, y, z)=0$ [$\varphi(x, y, z)=0$] be the equation of F [of Φ]. Put $J_x=\partial(f\varphi)/\partial(yz)$, $\cdots$. Let $\alpha(x, y, z)$ denote the general polynomial of degree $m+n-2$ which vanishes at the P_j's and put

$$\mathfrak{J}=\int_\Gamma \frac{\alpha}{z^2}\frac{dz}{J_z}=\int_\Gamma \frac{\alpha}{z^2}\frac{dx}{J_x}=\cdots.$$

From now on assume that Γ has no multiple points. Then $\mathfrak{J}$ has poles with the residues

$$r_j=\left\{\frac{\alpha_x J_x+\cdots}{J_z^2}\right\}_{P_j}$$

at the P_j's and no poles with residues $\neq 0$ elsewhere. The relation $\sum_1^{m\cdot n} r_j=0$ then leads (i) to the Jacobi conditions for the two plane curves $f(x, y, 0)=0$ and $\varphi(x, y, 0)=0$ to have the intersection G_{mn} and (ii) to the required conditions

$$(1)\qquad \sum_j \{A\cdot(f_xJ_x+f_yJ_y)\cdot J_z^{-2}\}_{P_j}=0,$$

$$(2)\qquad \sum_j \{B\cdot(\varphi_xJ_x+\varphi_yJ_y)\cdot J_z^{-2}\}_{P_j}=0,$$

between the tangents of Γ at the P_j's. Here $A=A(x, y)$ [$B=B(x, y)$] is the general polynomial of degree $n-2$ [$m-2$]. Thus (1) [(2)] stands for a system of $\binom{n}{2}\left[\binom{m}{2}\right]$ conditions. There are exactly $\binom{m-n}{2}$ relations between them.

P. Scherk (Saskatoon, Sask.).

Igusa, Jun-ichi. Analytic groups over complete fields. Proc. Nat. Acad. Sci. U. S. A. 42 (1956), 540–541.

It is shown that a commutative analytic group over a complete field k (of characteristic 0) is locally isomorphic with k^n.

The author derives from this result a theorem due to Mattuck [Ann. of Math. (2) 62 (1955), 92–119; MR 17, 87].

W. T. van Est (Utrecht).

Igusa, Jun-ichi. Fibre systems of Jacobian varieties. Amer. J. Math. 78 (1956), 171–199.

Let V be a nonsingular surface in a projective space. It is shown that there exists a general linear pencil $\{C_u\}$ on V; this is a linear system of hypersurface sections, parametrized by the variable point u of a projective line, such that each curve C_u is irreducible and in addition is nonsingular except for a finite number of u's, in which case its only singularity is one ordinary double point, and any two curves of the pencil are transversal on V at each base point of $\{C_u\}$. Next, the generalized jacobian variety [Rosenlicht, Ann. of Math. (2) 59 (1954), 505–530; MR 15, 823] of a curve C in projective space is constructed by the method of Chow [Amer. J. Math. 76 (1954), 453–476; MR 15, 823]; this method not only avoids extending the ground field but also gives a specific projective embedding. If C has no singularity except for one ordinary double point, the completion of its generalized jacobian is obtained by adding a copy of the ordinary jacobian variety of C as a "double variety." Now let V and $\{C_u\}$ be as before, let k be an algebraically closed field of definition for everything, let u be generic over k, and let the jacobian variety J_u of C_u and the canonical map φ_u: $C_u\rightarrow J_u$ (normalized so as to send a given base point of $\{C_u\}$ into 0) be constructed as above, all defined over $k(u)$. The "Néron Variety" $\mathcal{J}$ of V and $\{C_u\}$ is then the locus over k of $u\times J_u$. For any specialization $u\rightarrow u'$, J_u specializes into the jacobian $J_{u'}$ of $C_{u'}$ (or the completion of the generalized jacobian of $C_{u'}$ if $C_{u'}$ has a double point), the group law on J_u specializes to that on $J_{u'}$, and there is a map φ: $V\rightarrow\mathcal{J}$ (birational if the genus g of C_u is >0) reducing to φ_u on C_u. Furthermore, the singular locus of $\mathcal{J}$ is contained in its degenerate fibres, i.e. in the fibres corresponding to curves with double points. The map φ gives an isomorphism between the Albanese varieties of $\mathcal{J}$ and of V, and an injective mapping of differentials of the first kind on $\mathcal{J}$ into those on V. An appendix computes the linear equivalence class of the hyperplane sections of Chow's projective embedding of the jacobian J of a curve C; this class is algebraically equivalent to a multiple of the canonical image Θ in J of the $(g-1)$-fold product of C with itself.

M. Rosenlicht (Evanston, Ill.).

Igusa, Jun-ichi. Fibre systems of Jacobian varieties. II. Local monodromy groups of fibre systems. Amer. J. Math. 78 (1956), 745–760.

This paper begins the study of an algebraic analogue of the Poincaré normal functions [cf. Zariski, Algebraic surfaces, Springer, Berlin, 1935]. There are a number of interesting special results, but we confine ourselves to the main ideas: Using the notations of the previous review, let a be a quantity such that the fibre C_a has a double point. For any integer n prime to the characteristic p of k, the groups of points of order n of J_u and the generalized jacobian J_a' of C_a are the direct products of cyclic groups of order n, in number $2g$ and $2g-1$, respectively. Imagining the formal power series field $k((u-a))$ to be embedded in the universal domain, it turns out that the points of order n of J_u that specialize (over the natural k-specialization of $k((u-a)))$ to points of J_a' are all rational over $k((u-a))$; these points form the group of invariant points of order n, isomorphic to the direct product of $2g-1$ cyclic groups of order n. The group of vanishing points of order n is the subgroup of points that specialize into points of the one-parameter rational subgroup of J_a'; this is cyclic of order n. There is a pairing of the entire group of points of order n of J_u with itself into the group of nth roots of unity such that the subgroups of invariant and vanishing points are the annihilators of each other. Now let $n=l^\nu$, where $l\neq p$ is prime, let $\nu\to\infty$, and consider inductive limits. The points of J_u of order a power of l form a group isomorphic to the $2g$-fold direct product of the additive group of l-adic numbers mod 1. The $k(u)$-automorphisms and the $k((u-a))$-automorphisms of the universal domain therefore have representations as matrices of degree $2g$ whose coefficients are l-adic integers, giving the abstract monodromy group and abstract local monodromy group of the fibre system at l. The latter group is isomorphic to an additive group of l-adic integers. *M. Rosenlicht* (Evanston, Ill.).

★ **Weil, André. On the projective embedding of Abelian varieties.** Algebraic geometry and topology. A symposium in honor of S. Lefschetz, pp. 177–181. Princeton University Press, Princeton, N. J., 1957. $7.50

It is proved here that each abstract Abelian variety A is biregularly embeddable into a projective space. [For abstract Abelian varieties, see A. Weil, Variétés abéliennes et courbes algébriques, Hermann, Paris, 1948; MR 10, 621.] In addition, the following precise theorem is proved: Let X be a positive divisor on A. In order that there may exist an integer $n>0$ such that the class of nX be ample on A, it is necessary and sufficient that X should be non-degenerate. Here, following a paper by Morikawa [Nagoya Math. J. 6 (1953), 151–170; MR 15, 464], a divisor X on A is called non-degenerate if there are only finitely many points t of A such that $X_t\sim X$, where X_t denotes the transform of X by the translation $x\to x+t$, and where $\sim$ means "linearly equivalent". The proof of the above theorem is very simple if one takes the results of the author's book (cited above) for granted. *P. Roquette* (Hamburg).

★ **Chow, Wei-Liang. On the projective embedding of homogeneous varieties.** Algebraic geometry and topology. A symposium in honor of S. Lefschetz, pp. 122–128. Princeton University Press, Princeton, N. J., 1957. $7.50.

This article is a complement to one by A. Weil in the same volume on pp. 177–181 [see the paper reviewed above]. Weil proves that every Abelian variety (over any field) is embeddable in projective space. The present author shows that the same result holds for any homogeneous variety (not necessarily complete). In only two respects does Weil's proof need any modification. First, if the variety is not complete, the linear system needed for the embedding may be infinite-dimensional. This difficulty is circumvented by using a lemma which asserts that any abstract variety V is the image under a regular birational map of an open set W in a complete projective variety V'. One then uses linear systems on V' instead of V. Secondly, one has to extend the following lemma from the theory of Abelian varieties: If X is any divisor on an Abelian variety A and X_a the divisor obtained from X by the translation a, then $3X\sim X_a+X_b+X_{a^{-1}b^{-1}}$. This is easily done using the theory of Picard varieties (applied to the projective variety V').

It is perhaps worth pointing out that the first of the two lemmas mentioned above has proved a useful tool in extending to abstract varieties results known for projective varieties. This is particularly true when V is complete; in this case $W=V'$. *M. Atiyah.*

See also: Snapper, p. 868; Terpstra, p. 868; Nagata, p. 869; Cartier, p. 870.

NUMERICAL ANALYSIS

Numerical Methods

★ **Collatz, L. Fehlermaszprinzipien in der praktischen Analysis.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 209–215. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

Several criteria are summarized for approximating a function by a linear combination of given functions, and applications are described briefly, mainly to the solutions of partial differential equations. *A. S. Householder.*

Cooke, J. C. Osculatory interpolation and integration. J. Math. Phys. 35 (1957), 394–400.

"Formulas are given for interpolation and integration of a function of x, which is tabulated, together with its first, or its first two, or its first three derivatives, at equal intervals of the argument x." The treatment is not general, but a number of particular formulas are derived, using symbolic methods. The author states "The aim will be to push the formulas as far as possible towards the higher derivatives, as this will lead to higher accuracy. We shall not take differences, beyond the first, of the function itself or of its intermediate derivatives."

T. N. E. Greville (Washington, D.C.).

Lotkin, Mark. Note on the sensitivity of least squares solutions. J. Math. Phys. 35 (1956), 309–311.

This note gives formulae for determining the effect, on a least squares solution and on the residual vector, of an error in one observation. An example illustrates the method, which seems convenient only in the special case treated, notwithstanding the claim that "the answer can be supplied readily for the general case". *L. Fox.*

Luke, Yudell L. On the computation of log Z and arc tan Z. Math. Tables Aids Comput. **11** (1957), 16–18.

In a previous note, Clenshaw [MTAC **8** (1954), 143–147; listed in MR **16**, 128] has given numerical values of coefficients for the expansion of some transcendental functions in Chebyshev polynomials. In particular, he tabulates the coefficients for $\log(1+x)$ and arc tan x to nine decimal places. Here, treating these functions in a more general form, we determine precise theoretical coefficients and show that the development leads to formulas for computation in the complex domain.

Author's summary.

Lyusternik, L. A. Solution of problems of linear algebra by the method of continued fractions. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. no. **2** (1956), 85–90. (Russian)

The basic problem is to make more precise (i.e., accelerate the convergence of) the ordinary iterative algorithm $b^{(k)}=Ab^{(k-1)}+b$ for solving the finite linear system (*) $y=Ay+b$. Here A is a symmetric matrix of order n, and b is assumed to belong to no invariant space of A of dimension $<n$.

Let $R_0(\lambda)=\sum_{s=0}^{\infty} d_s/\lambda^{s+1}$, where $d_s=(A^sb, b)$. Let $q_k(\lambda)$ be the denominator of the kth convergent to the continued fraction

$$R_0(\lambda)=\frac{\lfloor \alpha_0}{\lambda-\beta_0|}-\frac{\lfloor \alpha_1}{\lambda-\beta_1|}-\cdots.$$

Write $q_k(\lambda)=a_0\lambda^k+a_1\lambda^{k-1}+\cdots+a_k$, and $a_s'=a_s/q_k(1)$. The author's solution to his problem is to approximate y by the following formula, exact when $n_0=n$:

(**) $y=$

$$b^{(k_0)}-\Big(\sum_{s=0}^{n_0-1}a_s'\Big)b^{(k_0)}+\Big(\sum_{r=1}^{n_0-1}a'_{n_0-r-1}b^{(k_0+r)}+a'_{n_0-1}b^{(k_0+n_0)}.$$

There is a discussion of how to find the $\{a_s\}$ in principle from the $\{d_i\}$, but the process is not reduced to an algorithm. There is a reference to the work of Lanczos [J. Res. Nat. Bur. Standards **45** (1950), 255–282; MR **13**, 163] for getting the $\{a_s\}$. {However, there is no mention of the well developed QD algorithm of Rutishauser [e.g., Z. Angew. Math. Phys. **5** (1954), 496–508; MR **16**, 863] for the same purpose. Nor is the work of Hestenes and Stiefel [J. Res. Nat. Bur. Standards **49** (1952), 409–436; MR **15**, 651] mentioned, although their conjugate-gradient algorithm apparently goes directly from the equation (*) to the solution (**) for $k_0=0$, avoiding the computation of the $\{d_i\}$ and the $\{a_i\}$, at least when $A-I$ is a definite matrix.}

G. E. Forsythe (Stanford, Calif.).

Mackenzie, J. K. A least squares solution of linear equations with coefficients subject to a special type of error. Austral. J. Phys. **10** (1957), 103–109.

Following the classical method a least squares solution is given for the equations $\sum_{k-1}^{s}(a_{rk}+e_rb_{rk})x_k=e_r$ $(r=1, \cdots, n\geqq s)$, where the a_{rk} and b_{rk} are fixed known constants and the e_r are observed values subject to error. The solution is obtained as a series in the successive moments of the joint distribution of the e_r, and only terms up to those involving the variance are retained. In this approximation the estimated values of the x_k are biased, but, after correction for this bias and using a particular weight for each equation, the classical tests of significance for the case $b_{rk}=0$ can be applied unchanged. With suitable assumptions it is shown that the series converges more and more rapidly as $n\to\infty$ for almost all sequences of the e_r. (Author's abstract.)

A. S. Householder.

Roth, J. P.; and Scott, D. S. A vector method for solving linear equations and inverting matrices. J. Math. Phys. **35** (1956), 312–217.

The vector method described is an extension of that of Purcell [same J. **32** (1953), 180–183; MR **15**, 47)] and removes some of the deficiencies of that process at the expense of a substantial increase in the number of necessary multiplications. A numerical example with five equations shows in detail the calculations involved, and the corresponding process for inverting a matrix of order three is explained fully. Certain advantages are claimed for the method, including economisation of storage, ease of scaling and the avoidance of divisions.

L. Fox.

Blanc, Ch. Sur le calcul approché d'une dérivée. Bull. Tech. Suisse Romande **83** (1957), 109–113.

This paper contains a discussion of the dangers of evaluating derivatives through use of interpolating polynomials and proposes use of least square principles to obtain better results. The argument proceeds by examples and no general results are derived.

P. C. Hammer (Madison, Wis.).

Stein, P. A note on numerical integration. Math. Gaz. **40** (1956), 268–270.

The author presents a method of integrating an arbitrary polynomial $f(x)$. This polynomial is expanded in a Taylor series which is convergent in the range $-1\leqq x\leqq 1$. By means of simple manipulation, the author obtains. Simpson's 1/3 rule, Tchebychef's formula and Tupper's formula for degree five and a much simpler one for degree seven from Tupper's paper [Math. Gaz. **39** (1955), 209–210].

H. Saunders (Philadelphia, Pa.).

O'Beirne, Thos. H. Can numerical integration be exact? Math. Gaz. **41** (1957), 59–60.

A discussion of the application of the corrected trapezoidal rule of Gauss to the Fourier analysis of a discrete set of equidistant ordinates, i.e. to trigonometrical interpolation.

Winn, E. A. A matrix method for the numerical solution of linear differential equations with variable coefficients. J. Roy. Aero. Soc. **61** (1957), 133–134.

The basis of the method is an integrating matrix $Q=(a_{ik})$ which may be composed from a number of previously chosen quadrature formulas of the form

$$y_i=y_0+h\sum_{k=0}^{n}a_{ik}y_k'+O_i(h^m)\quad(i=0, 1, \cdots, n)$$

with n denoting the number of points at which a solution is desired. The repeated differentiation of these relationships, and their subsequent insertion into the differential equations results in a system of linear algebraic equations of order $n+1$, from which the values of the highest order derivative occurring in the problem may be obtained in terms of y_0 and y_0' by matrix inversion. Knowing the values of the highest derivative the lower derivatives as well as the desired function itself are then determined quite easily. The method is illustrated by means of a second order linear differential equation with prescribed boundary conditions.

The procedure may be of utility in cases where the number n of points at which solutions are desired is reasonable.

M. Lotkin (Stratford, Conn.).

Mel'nik, S. I. The principle of St. Venant and oscillating functions. Uspehi Mat. Nauk (N.S.) 12 (1957), no. 1(73), 218–222. (Russian)

In a previous paper [Dokl. Akad. Nauk SSSR (N.S.) **95** (1954), 705–708; MR **16**, 79] the author defined oscillating functions, and applied them to the approximate solution of integral equations, with error bounds. Here oscillating functions of order m are defined and application is made to the solution of ordinary differential equations and to Saint Venant's principle.

A. S. Householder (Oak Ridge, Tenn.).

Ionescu, D. V. Généralisation d'une propriété qui intervient dans la méthode de Runge et Kutta d'intégration numérique des équations différentielles. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **8** (1956), 67–100. (Romanian. Russian and French summaries)

Dans ce travail, l'auteur étend aux systèmes d'équations différentielles et aux équations différentielles d'ordre supérieur le procédé numérique d'intégration, indiqué dans un travail antérieur [même Bull. **6** (1954), 229–241; MR **16**, 1158]. *D. S. Mitrinovitch* (Belgrade).

Tihonov, A. N.; and Samarskiĭ, A. A. On finite difference schemes for equations with discontinuous coefficients. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 393–396. (Russian)

A numerical method for the approximation of the solution of a class of boundary value problems for linear ordinary differential equations with discontinuous coefficients is considered. A method for replacing derivatives by differences is given together with necessary conditions for determining the difference operator so that the solution of the difference equation should converge to the solution of the differential equation. *C. Saltzer.*

Mikeladze, Š. E. Numerical solution of the inhomogeneous polyharmonic equation. Inžen. Sb. **23** (1956), 190–202. (Russian)

The solution of the boundary value problem for the polyharmonic equation is discussed by the method of various types of grids. The existence and uniqueness of the solution of the linear system of simultaneous equations which are used to replace the partial differential equation are proved and the convergence of the solution of the approximate problem to the exact solution is discussed.

C. Saltzer (Syracuse, N.Y.).

Babuška, I.; and Mejzlík, L. Solution of partial differential equations by the method of nets. Časopis Pěst. Mat. **80** (1955), 331–358. (Czech)

An expository paper with an extended (13 pages) bibliography.

See also: Guest, p. 962; Laudet, p. 968; Diamantides and Horowitz, p. 978.

Graphical Methods, Nomography

Ballantine, J. P. Graphical solution of linear differential equations. Amer. Math. Monthly **64** (1957), 357–359.

The author writes the equation $y'+P(x)\cdot y=Q(x)$ in the form $y'=(B(x)-y)/(A(x)-x)$ and presents a procedure for the graphical solution of it based on the fact that y' is the slope of the straight line through the two points $(A(x),B(x))$ and (x, y), the initial point being (x_0, y_0). *S. Kulik.*

Smirnov, S. V. On the improvement of an approximate nomogram by the shifting of scales. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki. **10** (1956), 80–96. (Russian)

Tables

Brundell, P.-O. A new table of the amplitude functions of the iterated sine- and cosine-integrals and some comments on the aperiodic functions in Hallén's antenna theory. Kungl. Tekn. Högsk. Handl. Stockholm no. 108 (1957), 14 pp.

The second part of the title refers to a possible way of establishing representation and uniqueness theorems of $f(x)=\sum_{m=1}^{n}\phi_m(x)e^{i\lambda_m x}$, in which λ_m is real and $\phi_m(x)$ denotes an "aperiodic" function characterized by certain properties of monotony. The first part refers to six-decimal values of 16 functions defined by definite integrals occurring in antenna theory, computed on the BESK.

C. J. Bouwkamp (Eindhoven).

Machines and Modelling

Bennett, J. M. Digital computers and the load-flow problem. Proc. Inst. Elec. Engrs. B. **103** (1956), supplement no. 1, 16–25.

Describes the use of a digital computer for solving power distribution problems of the type usually solved on network analyzers. The solution has two stages. In the first the set of equations

$$P_r+jQ_r=I_r'(V_0'-V_r')^*$$

is solved by successive iteration for I_r', the set of currents extracted from the rth busbar, where P_r, Q_r are the real and reactive power extracted at the busbar, V_0' is the voltage above ground of a reference busbar, and V_r' the voltage drop relative to the reference. In the second stage V', the matrix of voltage drops across the network elements, is found from $V'=Y'^{-1}I'$, where Y' is the admittance matrix of the network, as computed by the machine.

The inversion method is that where Y' is first expressed as the product of lower and upper triangular matrices; in the computer program Y' is partitioned into 4×4 submatrices and the inversion carried out interpretively using these submatrices as blocks. The results on a network consisting of 22 busbars at 46 branches is given as an example. The paper concludes with some suggestions as to where the program might be improved, a comparison with the network analyzer, and some comments on computer design for this application. *C. C. Gotlieb.*

Davis, Philip. Numerical computation of the transfinite diameter of two collinear line segments. J. Res. Nat. Bur. Standards **58** (1957), 155–156.

This describes a controlled experiment in computing. The transfinite diameter τ of the set consisting of the pair of segments $(-1, -a)$, $(a, 1)$ where $0<a<1$ is known to be $\frac{1}{2}(1-a^2)^{\frac{1}{2}}$. It has been shown by M. Fekete and J. L. Walsh (J. Analyse Math., **4**, 49–87 (1955); MR **17**, 354) that the transfinite diameter of a set consisting of a finite number of Jordan arcs is $\lim(a_n)^{1/n}$ where $a_n(>0)$ is the leading coefficient of the polynomial $p_n(x)=a_nx^n+\cdots$, where the sequence $\{p_n\}$ is orthonormal over E. It follows

that the actual computation of the $p_n(x)$ will give some information on the effectiveness of orthonormalizing processes and on practicality of the Fekete-Walsh representation of τ. In practice, for $n=10$, with 8D floating arithmetic, about two place accuracy was achieved.
John Todd (Washington, D.C.).

de Finetti, Bruno. Gli strumenti calcolatori nella Ricerca Operativa. Civiltà delle Macchine **5** (1957), 18–21.

Il presente articolo costituisce il testo di una delle relazioni presentate al primo convegno tenutosi in Italia sulla Ricerca Operativa (a Torino, 5–6 dicembre 1956, per iniziativa della Unione Industriale che celebrava il proprio cinquantenario). Gli Atti, contenenti discorsi, relazioni e discussioni, sono in corso di stampa a cura della stessa Unione Industriale di Torino (20, via Massena).
Dal riassunto dell'autore.

Davies, D. W. Sorting of data on an electronic computer. Proc. Inst. Elec. Engrs. B. **103** (1956), supplement no. 1, 87–93.

Describes sorting of data stored on magnetic tapes by an electronic digital computer. A distinction is drawn between "true sorting" where items are distributed according to the keys on which they are being sorted, and "reordering" where there is merely a rearrangement of a sequence. After introducing basic terminology and some examples where sorting arises, the methods are presented. These are mainly variants of radix sorting, where the data is subdivided into groups according to the characters in the key, and merging, where strings of items are successively combined. Tape handling is discussed and some general comparisons of the different methods made.
C. C. Gotlieb (Toronto, Ont.).

Brooker, R. A. The programming strategy used with the Manchester university Mark I computer. Proc. Inst. Elec. Engrs. B. **103** (1956), supplement no. 1, 151–157.

The paper gives an account of the programming strategy developed for use with the University of Manchester electronic computer Mark I, a typical 2-level storage machine of which a brief description is included. The topics dealt with include: the method of storing and calling in routines; the representation of instructions outside the machine; the organization of the library of sub-routines; the mechanism of interpretive routines for double-length and floating-point operations; matrix operations; partial differential equations; an attempt at automatic programming; mistake diagnosis in programmes; and measures for dealing with machine breakdown. (From the author's Summary.) *C. C. Gotlieb.*

Murphy, R. W. A positive-integer arithmetic for data processing. IBM J. Res. Develop. **1** (1957), 158–170.

Author hypothesizes that positive numbers suffice for expressing quantities in accounting. He accomplishes this by defining the primary recursive operator diminish, θ, where $x\theta y=x-y$ if $x\geqq y$, $=0$ if $x<y$. It is proved that the diminish operator produces only continuous functions but the introduction of multiplication allows discontinuous quantities to be represented as well. Several examples are given to show how certain operations arising in data processing can be programmed more simply on a machine which has the diminish instruction in its order code than on one which has only the conventional arithmetic instructions. In the examples the use of the new operator produces a shorter code without control transfers, but one which would take more program steps to execute.
C. C. Gotlieb (Toronto, Ont.).

Denison, S. J. M.; and Taylor, D. G. The use of digital computers in obtaining solutions to electric-circuit problems involving switching operations. Proc. Inst. Elec. Engrs. B. **103** (1956), supplement no. 1, 35–46.

Programs are developed to perform a number of switching operations in succession. Illustrations of the method include the transient analysis of several power-rectifier circuits. *G. Kron* (Schenectady, N.Y.).

Haselgrove, C. B.; and Hoyle, F. A mathematical discussion of the problem of stellar evolution, with reference to the use of an automatic digital computer. Monthly Not. Roy. Astr. Soc. **116** (1956), 515–526 (1957).

The problem of stellar evolution is expressed, mathematically, by a set of non-linear partial differential equations describing the variation of density and temperature as a function of time and of distance from the star center. This problem is tackled in the paper under review on the following assumptions: i) Spherical symmetry of the star; ii) hydrostatic equilibrium; iii) negligible stirring of material except where convectively unstable; there, "well mixed".

The computer employed was one of only moderate storage and speed, viz. the Cambridge EDSAC I; thus, considerable ingenuity was required in defining subsidiary variables.

The following set of differential equations determines the structure of the star at time t, assuming its composition at time t is known as well as the course of the evolution preceding t:

$$\frac{dR}{dM}=\frac{1}{4\pi R^2\rho},\quad \frac{dP}{dM}=-\frac{GM}{4\pi R^4},\quad \frac{dQ}{dM}=U+\frac{\partial\epsilon}{\partial t},$$

$$\frac{dT}{dM}=\begin{cases}-\dfrac{3KQ}{64\pi^2ac\rho T^3R^4} & \left(\geq\dfrac{\Gamma-1}{\Gamma}\dfrac{T}{P}\dfrac{dP}{dM}\right),\\[2ex] \dfrac{\Gamma-1}{\Gamma}\dfrac{T}{P}\dfrac{dP}{dM} & \text{(otherwise).}\end{cases}$$

Here, the main variables are M the mass inside a spherical surface of radius R within the star, Q the energy flux across the spherical surface, P total pressure, and T temperature at radius R; and the subsidiary variables at radius R are ρ the density, K opacity, U rate of thermonuclear energy generation, ϵ potential energy per unit mass, and Γ Chandrasekhar's adiabatic exponent. Assuming now the structure at time t has been determined by the solution of these equations, the values of two variables $X(M, t+\delta t)$, $\beta(M, t+\delta t)$ can be found; here, X is the fraction by mass at radius R of the stellar material consisting of the particular element undergoing nuclear transformation and the parameter β takes positive integral values: $\beta=1$ if the main element subject to transformation is hydrogen, $\beta=2$ if it is helium, etc. for other values of β and other elements. The value of β serves to indicate to the computer which of a number of sub-routines is to be used in subsequent calculations. It is assumed, in determining $X(M, t+\delta t)$, $\beta(M, t+\delta t)$ that, for small δt, the structure worked out for time t can be taken as a good approximation over the whole time increment; i.e., X is a slowly varying function of M. These values of X, β now serve to determine the structure for time $t+\delta t$. Given the composition of the star at the mo-

ment of its origin, the evolution can thus be followed. The Runge-Kutta method of integration is used to integrate equations of the form

$$\frac{dy_i}{dt}=f_i(y_1, y_2, \cdots, y_n).$$

Results will be discussed in later papers.
W. F. Freiberger (Providence, R.I.).

Gilmour, A. The application of digital computers to electric traction problems. Proc. Inst. Elec. Engrs. B. **103** (1956), supplement no. 1, 59–67.

Schlitt, Herbert. Lösung einer Wärmeleitungsaufgabe durch Analogiebetrachtungen. Arch. Electrotech. **43** (1957), 51–58.

The author compares two systems: (1) A blackened platinum disc on the end of a wire in a gas-filled glass vessel. The disc is heated by being placed at the focal point of an external lens system; the vessel and end of the wire are held at a fixed temperature. (2) A finite, induction-free, transmission line is fed from an emf source through a RC network, and terminated with a short. By suitable choice of input network the electrical system is an exact analogue of the thermal system. The equivalence of the various physical quantities of the analogy are adduced, and the known formal solution to differential equations for the two systems are compared with an approximate solution of the heat flow problem obtained previously.
J. G. L. Michel (Middlesex).

Fodor, G.; and Temes, G. Differentiating and integrating circuits. Acta Tech. Acad. Sci. Hungar. **16** (1957), 73–104. (German, French and Russian summaries)

This paper discusses well-known passive and active circuits for differentiation and integration. From these more complex circuits admitting of higher accuracy are developed. Examples of the results of applying these principles to practical circuits are given.
J. G. L. Michel (Middlesex).

See also: Hall, Marshall, Jr., p. 867; Maškovič, p. 977; Diamantides and Horowitz, p. 978; Baranov, p. 978.

PROBABILITY

★ **Blackwell, David. On a class of probability spaces.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability 1954–1955, vol. II, pp. 1–6. University of California Press, Berkeley and Los Angeles, 1956. $6.50.

The author defines a Lusin space to be a set together with a Borel field **B** of subsets, such that **B** is countably generated and such that every real **B**-measurable function has for range an analytic set. A probability measure on a Lusin space is proved to be perfect in the sense of Gnedenko and Kolmogorov [Limit distributions for sums of independent random variables, Addison-Wesley, Cambridge, Mass., 1954; MR **12**, 839; **16**, 52], and to have certain other properties of regularity. *G. A. Hunt.*

★ **Bochner, S. Stationarity, boundedness, almost periodicity of random-valued functions.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. II, pp. 7–27. University of California Press, Berkeley and Los Angeles, 1956. $6.50.

Consider the equation (*) $\Lambda x(t)=y(t)$, where $x(t)$ and $y(t)$ are functions with values in a Hilbert space and Λ is the operator

$$\Lambda x(t)\equiv \sum_{\rho=0}^{r} \frac{d^\rho x}{dt}(t-\tau)dC_\rho(\tau),$$

each function $C_\rho(\tau)$ being of bounded variation and the function $\Lambda(e^{i\alpha_t})$ of α being assumed twice differentiable. Suppose $x(t)$ to be a weak solution of (*) for a given function $y(t)$ that is stationary in the sense of Khinchin. The author proves, under various hypotheses of boundedness on $x(t)$ and the appropriate supplementary hypotheses on $\Lambda(e^{i\alpha t})$, the existence of another solution of (*) which is stationary. The paper is an enlarged and completed version of an earlier one [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 289–294; MR **15**, 807]. *G. A. Hunt.*

★ **Chung, K. L. Foundations of the theory of continuous parameter Markov chains.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. II, pp. 29–40. University of California Press, Berkeley and Los Angeles, 1956. $6.50.

Let x be a Markoff process on the integers having stationary transition probabilities that satisfy $p_{ij}(t)\to 0$ as $t\to 0$. The author proves that there is an equivalent process (on the Alexandroff compactification of the integers such that almost all sample functions have the following properties: Let S_i be the set of t for which the sample value is i; if i is stable, the closure $\bar{S}_i$ is the union of countably many disjoint closed intervals, only finitely many being included in any compact set; if i is instantaneous, $\bar{S}_i$ is perfect and nowhere dense. The set of discontinuities D is closed, and it is the union of the $\bar{S}_i$, as i ranges over the instantaneous states, and a set of measure zero included in the closure of the set of jumps. A complementary interval of D is included in some S_i, with i stable, and two such contiguous intervals correspond to distinct states. $\bar{S}_i$ differs from S_i by a countable set, and $\bar{S}_i\cap\bar{S}_j$ is at most countable if $i\neq j$. A number of similar statements are also proved. Although several of the results are not new [cf. especially P. Lévy, Ann. Sci. Ecole Norm. Sup. (3) **68** (1951), 327–381; MR **13**, 959], this is the first brief, precise treatment of the material. The results are used in the last section to discuss optional stopping and splitting of the process. The following corrections have been furnished by the author: (a) In stating (iii), p. 34, j must be allowed to have the value ∞; this is permissible, by p. 36, second paragraph. (b) On p. 39, replace $\{\Lambda;\ x(rh,\ \omega)\neq i,\ 0\leq r<m;\ x(mh,\ \omega)=i\}$ by $\{\Lambda;\ \alpha_n=mh\}$, where

$$\alpha_n=\inf(vh\colon vh>\alpha(\omega),\ x(vh,\ \omega)=i)\quad (h=2^{-n}).$$

G. A. Hunt.

★ **Copeland, A. H., Sr. Probabilities, observations and predictions.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. II, pp. 41–47. University of California Press, Berkeley and Los Angeles, 1956. $6.50.

The author has previously defined implicative Boolean algebras [Math. Z. **53** (1950), 285–290; MR **12**, 721] and has treated the foundations of probability in terms of them [Studies in mathematics and mechanics presented to Richard von Mises, Academic Press, New York, 1954, pp. 278–284; MR **16**, 376]. The treatment is amplified in

the present paper, and a method of testing hypotheses is sketched. *G. A. Hunt.*

★**Doob, J. L. Probability methods applied to the first boundary value problem.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. II, pp. 49–80. University of California Press, Berkeley and Los Angeles, 1956. $6.50.

Let R be a locally compact separable Hausdorff space, $\{D\}$ a collection of open subsets having the following properties. (a) Every open set is a union of sets D; every D has compact closure. (b) Associated with each pair (z, D), z in D, there is a measure $\mu(z, D, dy)$ of total mass 1 concentrated on the boundary D' of D. (c) The integral $\int \mu(z, D, dy)f(y)$ is continuous for z in D, if it is finite on a set dense in D and if f is positive; it is continuous on the closure of D if f is continuous. (d) If $D\cup D'\subset D_1$ and $z \in D$, then $\mu(z, D_1, E)=\int \mu(z, D, dy)\mu(y, D_1, E)$. (e) For each z in D there exist arbitrarily small $D_0, \cdots, D_n$ and points z_k in D_k such that $z_0=z$, $D_n\cap D'\neq\emptyset$, $z_k \in D_k$, z_{k+1} belongs to the support of $\mu(z_k, D_k, dy)$. The author proves some basic properties of (sub)regular functions, which are defined in terms of the sets D and the associated measures just as (sub)harmonic functions are defined in terms of solid spheres and harmonic measure. From now on assume R to be the union of an increasing sequence of sets D_n. (A much weaker hypothesis is used in the paper.) If z is a point of D_N, say, let (z_n) be a Markov process having transition measures $\mu(x, D_{N+n}, dy)$, starting at z, and defined over some probability space Ω. The sequence $u(z_n)$ is shown to be a (semi)martingale if u is (sub)regular, and $x(z, \omega)= \lim u(z_n(\omega))$ is proved to exist for almost all ω if u is suitably bounded. The notion of stochastic boundary function is defined, and a suitably bounded regular function is shown to be the expectation of a unique stochastic boundary function, which is precisely the limit mentioned in the preceding sentence. These results extend the ones obtained by the author for subharmonic functions [Trans. Amer. Math. Soc. **77** (1954), 86–121; MR **16**, 269] and subparabolic functions [ibid. **80** (1955), 216–280; MR **18**, 76]. Let a boundary R' be adjoined to R in some manner. The author discusses, in the present setting, the method of Perron-Wiener-Brelot [Brelot, Acta Litt. Sci. Szeged **9** (1939), 133–153; MR **1**, 121] for constructing a regular function related to a given function f on R'. Only the most striking result will be mentioned: Let f be PWB resolutive, with u the corresponding regular function, let (z_n) be one of the Markov processes mentioned above, and let $R(\omega)$ be the set of limit points of the sequence $(z_n(\omega))$ belonging to R'; it is proved that u is the expectation of its stochastic boundary function $x(z, \omega)$, and that f has the constant value $x(z, \omega)$ on almost every set $R(\omega)$. The PWB method is shown to be strengthened if topological approach to the boundary R' is replaced by approach along the trajectories of the processes (z_n). [The author has informed me that the last paragraph of § 6 should be ignored.] *G. A. Hunt.*

★**Fortet, Robert M. Random distributions with an application to telephone engineering.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. II, pp. 81–88. University of California Press, Berkeley and Los Angeles, 1956. $6.50.

The author considers the flow of calls in a telephone exchange, using simple stochastic integral equations to connect incoming traffic, holding times, and the numbers of calls lost, waiting, or in train. A sketch of I. M. Gelfand's note on random Schwarz distributions [Dokl. Akad. Nauk SSSR (N.S.) **100** (1955), 853–856; MR **16**, 938] is included. *H. P. McKean, Jr.* (Princeton, N.J.).

★**Hammersley, J. M. The zeros of a random polynomial.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. II, pp. 89–111. University of California Press, Berkeley and Los Angeles, 1956. $6.50.

The author gives a complete solution (at least in principle) to the problem of determining the distribution of the roots of the equation

$$g(z)=\sum_{j=0}^{n} c_j z^j=0$$

when the (complex) coefficients c_j have a given joint distribution. The first part of the paper is concerned with showing that if a sequence of distributions for the coefficients converges to a limiting distribution, then the sequence of corresponding distributions for the roots converges to the corresponding limit. Formulae are then derived giving the density functions for various joint distributions of the roots, on the assumption that the coefficients have a distribution with a continuous density and moments of all orders. Since every distribution is a limit of such "well-behaved" distributions, this gives a solution to the general problem. The author defines the condensed distribution of the roots, which may be most briefly described as the measure which assigns to a set $1/n$ times the expected number of roots falling in the set. Explicit formulae for the condensed distribution of the roots are given for the case in which the coefficients have an arbitrary non-degenerate normal distribution in complex $(n+1)$-space, and for the case in which they have a non-degenerate normal distribution in real $(n+1)$-space. This last result generalizes a result of Kac [Bull. Amer. Math. Soc. **49** (1943), 314–320; MR **4**, 196] on the expected number of real zeros of a polynomial with independent identically normally distributed coefficients. *H. F. Trotter.*

★**Harris, T. E. The existence of stationary measures for certain Markov processes.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. II, pp. 113–124. University of California Press, Berkeley and Los Angeles, 1956. $6.50.

Let X be a set, B a countably generated Borel field of subsets of X, and $P(x, E)$ a Markov transition function defined for x in X, E in B. Let m be a σ-finite measure on B such that $x_n \in A$ infinitely often with probability 1, if $m(A)>0$ and if (x_n) is a Markov process with $P(x, E)$ for transition function. The author proves the existence of a measure Q on B satisfying

$$Q(E)=\int Q(dx)P(x, E), \quad m(E)>0\Rightarrow Q(E)>0,$$

and shows that Q is unique up to a constant multiplier. *G. A. Hunt.*

★**Itô, Kiyosi. Isotropic random current.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. II, pp. 125–132. University of California Press, Berkeley and Los Angeles, 1956. $6.50.

The standard model for homogeneous isotropic turbu-

lence in 3 dimensional euclidean space R^3 consists of a set W of admissible fluid flows, a function u on $[0, +\infty)\times W$ to vector fields $u(t, x, w)$: $x \in R^3$ describing the velocities of the flow w at time t, and a suitable Borel field of subsets of W with probabilities attached, such that the distribution of u is invariant under the substitutions

$$u(t, x, w)\to u(t, x+a, w) \quad (a \in R^3)$$

and

$$u(t, x, w)\to g^{-1}u(t, gx, w) \quad (g \in O^3),$$

in which O^3 is the group of euclidean rotations.

Ignoring the time t, it is clear that the correlation function

$$E(u(x^1, w), a^1)\,(u(x^2, w), a^2))=f(x^2-x^1, a^1, a^2) \qquad (x^1, x^2, a^1, a^2 \in R^3)$$

is invariant under O^3, and, using the known structure of such invariants, it is not difficult to reduce f and functions like it to a convenient canonical form in which x^2-x^1, a^1, a^2 figure via the basic invariants

$$\|x^2-x^1\|,\ (x^2-x^1, a^1),\ (x^2-x^1, a^2),\ \|a^1\|,\ (a^1, a^2),\ \|a^2\|$$

alone, a remark due to H. P. Robertson [Proc. Cambridge Philos. Soc. **36** (1940), 209–223; MR **1**, 286]. Since f is positive semi-definite, a fact that Robertson overlooked, the reduction to canonical form just described is not complete, and the final result in this direction is due to S. Itô [Nagoya Math. J. **2** (1951), 83–92; MR **12**, 724].

The author extends S. Itô's result from vector fields on R^3 to differential forms and currents of de Rham on R^n, introducing the counterpart of f and reducing it to canonical form. When $n=3$ and the current is a vector field, his canonical form for f looks a little different from S. Itô's: it reads

$$\int_{R^3}\exp\{2\pi i(x^2-x^1, x)\}\,(\theta, a^1)(\theta, a^2)d\theta A(dr) + \int_{R^3}\exp\{2\pi i(x^2-x^1, x)\,[(a^1, a^2)-(\theta, a^1)(\theta, a^2)]d\theta B(dr) + (a^1, a^2)C,$$

in which $(r, \theta)=x$ in spherical polar coordinates, A and B are finite and non-negative, $A(0)=0=B(0)$, and the constant C is nonnegative, the three terms reflecting the splitting of the vector field into whirl-free, source-free, and harmonic fields with cross-correlations $=0$.

H. P. McKean, Jr. (Princeton, N.J.).

★ **Loève, Michel. Ranking limit problem.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. II, pp. 177–194. University of California Press, Berkeley and Los Angeles, 1956. $6.50.

For each n let X_{nk} $(1\leq k\leq k_n)$ be independent random variables with distribution functions F_{nk}, and let $X_{n1}^*\leq\cdots\leq X_{nk_n}^*$ be the corresponding order statistics. The author studies the limiting distribution of the X_{nk}^* as $n\to\infty$, assuming the existence of distribution functions F_n such that $F_{nk}-F_n\to 0$ uniformly in k. For r fixed, the limit distribution of X_{nr}^* (if it exists) must have the form

$$F(x)=\int_0^{L(x)}\frac{t^{r-1}}{(r-1)!}e^{-t}dt,$$

with L positive, increasing, not necessarily finite; and the distribution of X_{nr}^* tends weakly to F if and only if $\sum_k F_{nk}\to L$. For $r_n\to\infty$ and $k_n-r_n\to\infty$, the limit distribution of $X_{nr_n}^*$ must have the form

$$G(x)=(2\pi)^{-\frac{1}{2}}\int_{g(x)}^{\infty}e^{-\frac{1}{2}t^2}dt,$$

with g decreasing and not necessarily finite; the distribution of $X_{nr_n}^*$ tends weakly to G if and only if

$$(r_n-\textstyle\sum_k F_{nk})[\sum_k F_{nk}(1-F_{nk})]^{-\frac{1}{2}}\to g.$$

Limiting joint distributions are also treated; the applications include several results of N. V. Smirnov [Trudy Mat. Inst. Steklov. **25** (1949); MR **11**, 605; **13**, 853].

G. A. Hunt.

★ **Lukacs, Eugene. Characterization of populations by properties of suitable statistics.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. II, pp. 195–214. University of California Press, Berkeley and Los Angeles, 1956. $6.50.

Consider statistics of a sample $X_1, X_2, \cdots, X_n$ of n independent observations from a population with distribution function F. Sometimes the form of F can be inferred from the odd fact about the distribution of suitable statistics: for example, if n is ≥ 2 and if

$$\bar{X}=n^{-1}(X_1+X_2+\cdots+X_n)$$

and $Q=(X_1-\bar{X})^2+(X_2-\bar{X})^2+\cdots+(X_n-\bar{X})^2$ are independent, then F is normal. The author has collected a large number of results of this sort and has added a useful bibliography.

H. P. McKean, Jr.

★ **Menger, Karl. Random variables from the point of view of a general theory of variables.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. II, pp. 215–229. University of California Press, Berkeley and Los Angeles, 1956. $6.50.

Continuing his studies of function and variable [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 956–961; MR **15**, 92], the author considers the notion of a random variable as it occurs in probability and statistics, stressing his view that the probabilist starts with a Borel field $\mathfrak{T}$ of subsets A of a space S with known probabilities $p(A)$ $(A \in \mathfrak{T})$ attached and considers functions f on S to R^1 with $f^{-1}(a, b] \in \mathfrak{T}$ $(-\infty\leq a<b<+\infty)$, while the statistician, confronted with the actual event $a<f<b$, is concerned with the field $\mathfrak{F}$ of Borel subsets of the range of f and with estimating the probabilities $p(f \in B)$ $(B \in \mathfrak{F})$.

H. P. McKean, Jr. (Princeton, N.J.).

Morimura, Hidenori. On a renewal theorem. Kōdai Math. Sem. Rep. **8** (1956), 125–133.

Continuing the work of Kawata [J. Math. Soc. Japan **8** (1956), 118–126; MR **18**, 75], the author proves the following theorem. Let S_n be the sum of n mutually independent random variables with strictly positive expectations. It is supposed that, when $n\to\infty$, M_n, the average of the means of the summands, converges, and that the second moments and variances of the summands also converge in the sense of Cesaro. Other conditions are imposed which restrict uniformly the magnitides of the summands. Then it is proved that the limit

$$\lim_{x\to\infty}\int_{-\infty}^{x}dx\sum_{1}^{\infty}\left(n-\frac{x}{M_n}\right)P\{x<S_n\leq x+h\}$$

exists, and its value is given explicitly.

J. L. Doob.

★**Doob, J. L. Present state and future prospects of stochastic process theory.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 348–355. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

Expository paper.

Prohorov, Yu. V. Convergence of random processes and limit theorems in probability theory. Teor. Veroyatnost. i Primenen. 1 (1956), 177–238. (Russian. English summary)

The author applies a metrization of the space of complete finite measures on an arbitrary complete separable metric space $\mathfrak{R}$ to the study of stochastic process measures. (It is supposed throughout that the closed subsets of $\mathfrak{R}$ are measurable, and that the measure of a measurable set is the supremum of the measures of its closed subsets.) Only a sampling of his many results and applications can be given here. Let μ_i be a measure on $\mathfrak{R}$, and let F^ε be the ε neighborhood of F. Consider the infimum of the values of ε for which both the inequality $\mu_1(F)<\mu_2(F^\varepsilon)+\varepsilon$ and this inequality with μ_1, μ_2 interchanged are valid for all closed sets F. The author shows that this number defines a distance between μ_1 and μ_2 under which the space of measures becomes a complete separable metric space. Convergence in this space is the usual weak (vague) convergence of measures, the standard convergence of distributions if $\mathfrak{R}$ is a Euclidean space and the measures are probability measures. It is shown that a family of measures is compact if and only if the measures are uniformly bounded and are uniformly small outside a sufficiently large compact set.

The purpose of the paper is to apply the above results to stochastic process measures, particularly those for which the sample functions are almost all either continuous or continuous except for discontinuities of the first kind. For this purpose the space $\mathfrak{R}$ is usually either the space $C[0, 1]$ of continuous functions on $[0, 1]$, with the metric of uniform convergence, or the space $D[0, 1]$ of functions continuous on $[0, 1]$ except for discontinuities of the first kind, for which a suitable metric is defined. However necessary and sufficient conditions for compactness of a family of measures on a Hilbert space are also found, in terms of the characteristic functionals of the distributions.

Necessary and sufficient conditions for compactness of families of probability measures on $C[0, 1]$ and $D[0, 1]$ are found. In the case of $C[0, 1]$, these are simply that the probability of the set of elements x in $C[0, 1]$ with $|x(0)|>M$ should go to 0 uniformly with $1/M$, and that the oscillation of a function in $C[0, 1]$ in an interval of length δ converges to 0 in probability, uniformly, when $\delta\to 0$. In particular, if Kolmogorov's condition

$$M|x(t_2)-x(t_1)|^a\leq K|t_2-t_1|^\alpha$$

(where the left side is the indicated expectation, and a, K, $\alpha-1$ are strictly positive) is satisfied for every member of a sequence of probability measures on $C[0, 1]$, the constants being the same throughout, then convergence of the sequence of finite dimensional distributions implies that of the sequence of measures on $C[0, 1]$.

For each n, let $\xi_{n,1}, \cdots, \xi_{n,k_n}$ be mutually independent random variables, and suppose that

$$0=t_{n,0}<\cdots<t_{n,k_n}=1.$$

Then the plane points $(t_{n,k}, \sum_{j=1}^k \xi_{n,j})$ determine, for each n, a function in $C[0, 1]$ by linear interpolation, a function in $D[0, 1]$ whose graph is made up of successive horizontal line segments from the given points. Thus, for each n, the random variables induce a probability measure P_n on $C[0, 1]$ and a probability measure P_n' on $D[0, 1]$. If the random variables have, for each n, mean 0 and finite variances, and if $t_{n,k}$ is the variance of $\sum_{j=1}^k \xi_{n,j}$, it is shown that the sequence of measures on $C[0, 1]$ converges to the Brownian motion (Wiener) measure on $C[0, 1]$ if the and only if Lindeberg condition is satisfied. This result generalizes one of Donsker's [Mem. Amer. Math. Soc. **6** (1951); MR **12**, 723]. If the summands are individually negligible as $n\to\infty$, if P is the distribution on $D[0, 1]$ corresponding to a process with independent increments and no fixed discontinuities, then P_n' converges to P if and only if there is convergence of the finite dimensional distributions and if certain other specified conditions are satisfied. If the limit process in question has stationary increments, and if $t_{n,k}=k/k_n$, these conditions reduce to those obtained by Skorohod [Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 364–367; MR **17**, 1096]. Let a, b be functions on $[0, 1]$, with $a<b$, $a(0)<0<b(0)$. Suppose that the above summands have zero means, that L_n is the sum of their third order absolute moments, and that $t_{n,k}$ is the variance of $\sum_{j=1}^k \xi_{n,j}$. Let H be the subset of $C[0, 1]$ with $a(t)\leq x(t)\leq b(t)$ on $[0, 1]$. Then it is shown that

$$|P_n(H)-W(M)|\leq\varepsilon(L_n)L_n^{1/4}\log^2 L_n,$$

where the first factor on the right goes to 0 with its argument. This result generalizes and strengthens one due to Chung [Trans. Amer. Math. Soc. **64** (1948), 205–233; MR **10**, 132].

Some of the results have been discussed by the author in earlier papers [Uspehi Mat. Nauk (N.S.) **8** (1953), no. 3(55), 165–167; and (with Kolmogorov), Bericht über die Tagung Wahrscheinlichkeitsrechnung Math. Statistik, Berlin, 1954, pp. 113–126; MR **15**, 237]. For another approach to the same problem, involving less stress on metric spaces, see the paper reviewed above.

J. L. Doob (Geneva).

Skorohod, A. V. Limit theorems for stochastic processes. Teor. Veroyatnost. i Primenen. 1 (1956), 289–319. (Russian. English summary)

Let X be a complete separable metric space, with distance function ρ, and let K be the space of functions from $[0, 1]$ to X, with one-sided limits at all points, continuous to the right in $[0, 1)$, continuous to the left at 1. Several non-metric topologies are defined in K. For example, under the J_1 topology, $x_n\to x$ if and only if there is a sequence of continuous functions λ_n mapping $[0, 1]$ onto itself in a 1-1 way such that $\rho[x_n(t), x(\lambda_n(t))]\to 0$ and $\lambda_n(t)\to t$, both uniformly. Throughout the following f is a function on K to the real line, continuous in the J_1 topology, N is any subset of $[0, 1]$ which is denumerable, dense in the interval, and contains the points 0, 1, $\{\xi_n(t), 0\leq t\leq 1\}$ is a stochastic process almost all of whose sample functions are in K. It is shown that the distribution of $f[\xi_n(\cdot)]$ goes to that of $f[\xi_0(\cdot)]$ when $n\to\infty$, for all f, if and only if there is convergence of the finite dimensional distributions of the $\xi_n(t)$ process to those of the $\xi_0(t)$ process, for t restricted to N, and if for every $\varepsilon>0$

$$\lim_{c\to 0}\limsup_{n\to\infty} P\{\Delta(c, \xi_n(t))>\varepsilon\}=0,$$

where
$\Delta(c, \xi_n(t)) =$

$$\sup_{t-c<t_1<t<t_2<t+c} \min(\rho[\xi_n(t_1), \xi_n(t)]; \rho[\xi_n(t), \xi_n(t_2)]).$$

If the conditions are satisfied, f need only be supposed continuous almost everywhere ($\xi_0(t)$ process measure) on K. Corresponding results are obtained for the other topologies. In the paper reviewed above Prohorov obtains analogous results by metrizing the sample function spaces and the spaces of measures. As an interesting tool, the author proves the following theorem. Suppose that the finite dimensional distributions of the $\xi_n(t)$ process converge to those of the $\xi_0(t)$ process. It is proved that there is then a sequence of stochastic processes $\{x_n, 0\leq t\leq 1\}$, defined on the measure space consisting of $[0, 1]$ with Lebesgue measure, such that the finite dimensional distributions of the $x_n(t)$ and $\xi_n(t)$ processes are the same for each n, that the sample functions of the $x_n(t)$ process are almost all in K, and that $x_n(t)\to x_0(t)$ with probability 1 for t in N. *J. L. Doob* (Geneva).

Šaragina, Z. I. Local limit theorems for certain schemes of cyclic processes. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 521–522. (Russian)

The author states various theorems obtained in a dissertation. Let $P_k(t)$ denote the probability that an event E occurs k times in the interval $(0, t)$. Theorem: if the distribution of the recurrence times of E has mean μ, variance σ^2, and a finite third absolute moment, then uniformly in t as $k\to\infty$ we have

$$P_k(t) = \frac{\mu}{\sigma\sqrt{(2\pi k)}}\exp\frac{-(t-k\mu)^2}{2k\sigma^2}+o\left(\frac{1}{\sqrt{k}}\right).$$

Here the time may be either continuous or discrete. The corresponding integral limit theorem was obtained by W. Feller [Trans. Amer. Math. Soc. **67** (1949), 98–119, Th. 5, p. 105; MR **11**, 255]. The author states two other local limit theorems and also the following integral limit theorem: let $F(x)$ be the cumulative distribution function of recurrence times of E, and suppose that

$$\{1-F(\alpha x)\}/\{1-F(x)\}$$

tends to 1 as $x\to\infty$ for any $\alpha>0$. Then for the number N_t of occurrences of E in the interval $(0, t)$ we have

$$\Pr[N_t<z/\{1-F(t)\}]\to 1-e^{-z}(t\to\infty)$$

for any $z>0$. This contradicts an assertion that Doeblin's condition is necessary as well as sufficient for convergence to a non-normal limit law [cf. ibid., Theorem 6, p. 106]. *H. P. Mulholland* (Birmingham).

Parzen, Emanuel. A central limit theorem for multilinear stochastic processes. Ann. Math. Statist. **28** (1957), 252–256.

In a recent paper [Proc. Cambridge Philos. Soc. **49** (1953), 239–246; MR **14**, 771] Diananda proved a central limit theorem for discrete parameter stochastic processes which are linear. In this paper a central limit theorem is proved for a class of stochastic processes called multilinear by the author. *B. Epstein* (Detroit, Mich.).

Baxter, Glen; and Donsker, M. D. On the distribution of the supremum functional for processes with stationary independent increments. Trans. Amer. Math. Soc. **85** (1957), 73–87.

A separable stochastic process $\{x(t), 0\leq t<\infty\}$ with stationary increments whose sample functions vanish at the origin is characterised by $E\{e^{i\xi x(T)}\}=e^{T\psi(\xi)}$, where $e^{\psi(\xi)}$ is the Lévy-Khinchine representation of the characteristic function of an infinitely divisible distribution. The authors are concerned with the supremum functional and its distribution function $\sigma(\alpha, T)=P\{\sup_{0\leq t\leq T}x(t)<\alpha\}$. Under certain minor restrictions on the function $\psi(\xi)$, they find formulas expressing the double Laplace transform of $\sigma(\alpha, T)$ in terms of $\psi(\xi)$. This is done by first approximating the original process by a discrete one and then using an identity due to Spitzer [same Trans. **82** (1956), 323–339; MR **18**, 156]. Finally, several examples, some of them new, are treated. Wherever possible, the double Laplace transform is inverted giving $\sigma(\alpha, T)$ explicitly. *G. Hufford* (Stanford, Calif.).

Adhikari, Bishwanath Prosad. Quelques propriétés des processus stochastiques localement continus en probabilité. C. R. Acad. Sci. Paris **244** (1957), 1000–1002.

The author studies stochastic processes which are locally continuous in probability. The main result is stated in Theorem 2 which says the following. Let two probability measures, absolutely continuous with respect to each other, be defined on the sample space. If all finite dimensional distributions are absolutely continuous with respect to Lebesgue measure, form the likelihood ratio. Under the above assumptions this ratio converges almost certainly to the Radon-Nikodym derivative. This result is of importance in the theory of testing hypothesis for stochastic processes. — Theorem 4 is false as can be shown by counter examples. *U. Grenander* (Stockholm).

Ghermănescu, M. Sur les chaînes de Markov. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **8** (1956), 101–114. (Romanian. Russian and French summaries)

The author returns to the problem of determining the most general matrix $P(x)=(f_{jk}(x))$, where $f_{jk}(x)$ are functions of the real variable x, which verifies the functional equation

$$P(x+y)=P(x)P(y)$$

considering also the singular case. The results are known: Every element of P has the form $\sum_n Q_n(x)e^{a_n x}$, where $a_1, a_2, \cdots, a_m$ are constants and $Q_k(x)$ polynomials in x of determined degrees. *O. Onicescu* (Bucarest).

Anderson, T. W.; and Goodman, Leo A. Statistical inference about Markov chains. Ann. Math. Statist. **28** (1957), 89–110.

This paper considers a situation rather different from that examined by Bartlett [Proc. Cambridge Philos. Soc. **47** (1951), 86–95; MR **12**, 512] and others, in that the chain may not be time-homogeneous, and several simultaneous samples are available rather than one long one (although the latter case is discussed). The maximum likelihood estimates of the transition probabilities and their asymptotic distribution are calculated. Likelihood ratio tests and tests of chi-squared type are presented for the hypotheses (1) that the chain is time-homogeneous, (2) that the transition probabilities have specified values, and (3) that the chain is of a given order. Several other topics are discussed, among them the relation between the two types of test. *P. Whittle* (Wellington).

✶ **Сираждинов, С. Х. [Siraždinov, S. H.] Предельные теоремы для однородных цепей Маркова. [Limit theorems for stationary Markov chains.]** Izdat. Akad. Nauk Uzbekskoĭ SSR, Taškent, 1955. 84 pp. 3.50 rubles.

The author considers Markov chains with stationary

transition probabilities and states $e_1, e_2, \cdots, e_s$, the time being either discrete or continuous. Introduction: Summary of previous work on the asymptotic behaviour of Markov chains; enunciation of the author's principal results, mostly local limit theorems giving asymptotic expansions for the distribution of the vector whose αth component is the total time spent in state e_α during an interval $(0, t]$. Chapter I: The characteristic matrix. Lemmas on the latent roots of this matrix. (For discrete time the matrix is simply $(p_{\alpha\beta} \exp i\theta_\alpha)$, $\alpha, \beta=1, 2, \cdots, s$, where $(p_{\alpha\beta})$ is the usual matrix of transition probabilities.) Chapter II [III]: Limit theorems for the discrete [continuous] case. Theorem 1 generalizes a local limit theorem due to Kolmogorov [Izv. Akad. Nauk SSSR. Ser. Mat. **13** (1949), 281–300; MR **11**, 119]. Most of the theorems proved here had been announced previously by the author [Dokl. Akad. Nauk SSSR (N.S.) **84** (1952), 1143–1146, **98** (1954), 905–908; MR **14**, 187; **16**, 494]. One of them generalizes a result found by Sarymsakov [Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **5** (1949), 61–69; MR **16**, 495]. Many of the author's proofs make use of limit theorems given by Esseen [Acta Math. **77** (1945), 1–125; MR **7**, 312]. His exposition is straightforward and not hard to read, but there are many misprints, in particular, 0 for O or o. *H. P. Mulholland* (Birmingham).

Gihman, I. I. On asymptotic properties of certain statistics similar to χ^2. Teor. Veroyatnost. i Primenen. 1 (1956), 344–348. (Russian. English summary)

A sequence is considered of series of tests (independent in each series) which have possible outcomes $E_1, E_2, \cdots, E_n$ occurring with a probability of $p_1, p_2, \cdots, p_n$ respectively, where $p_j>0$ and $\sum_i p_i=1$. A group of possible outcomes $(E_{i_1}, E_{i_2}, \cdots, E_{i_m})$ are distinguished for which $\lim_{N\to\infty} \max_{1\leq s\leq m} p_{i_s}=0$, and $\sum_{k=1}^m p_{i_k}=\alpha_0$, where m and α_0 are independent of the number of series N. Theorems are given for sequences of series of certain statistics similar in structure to χ^2, which show that these sequences converge to corresponding continuous Markov processes. (Author's summary.) *J. L. Snell* (Hanover, N.H.).

Hida, Takeyuki. On the transition probability of a renewal process. Nagoya Math. J. **11** (1957), 41–51.

Let $X_1, X_2, \cdots$ be mutually independent non-negative random variables with a common distribution function. Let $n(t)$ be the number of sums $S_k=\sum_1^k X_j$ less than t, and let $x(t)=t-S_{n(t)}$. Then $x(t)$ has a simple interpretation in renewal theory. The $x(t)$ process was proved by the reviewer to be a Markov process with stationary transition probabilities [Trans. Amer. Math. Soc. **63** (1948), 422–438; MR **9**, 598]. Let P be the transition probability function of this process. The function P can be written down explicitly in terms of the distribution function of X_1. The author finds, under suitable hypotheses on the distribution function of X_1, differential equations satisfied by P, and shows that they determine P. *L. J. Doob.*

Le Roy, Jean. Formules matricielles du calcul du délai d'attente dans le cas des appels desservis au hasard. Ann. Télécommun. **12** (1957), 2–19.

The author re-examines, using matrix calculus, the problem of approximating the delay curves for calls served at random in a simple telephone system consisting of a simple trunk group with exponential holding (service) time distribution before which calls arrive at random, the subject of a paper by the reviewer [Bell System Tech. J. **32** (1953), 100–119; MR **14**, 664]. Moments of the delay distribution are obtained in matrix form as are coefficients in the MacLaurin series expansion of its complement, and are shown to be in agreement with the reviewer's results. The approximate delay curve is taken in the form $e^{-z}\sum K_n P_n(z)$, with the sum finite, $P_n(z)$ a normalized Laguerre polynomial, and the K_n determined in part by moments and in part by MacLaurin series coefficients, thus forcing a certain degree of contact with the exact curve near the origin. There are numerous misprints. *J. Riordan* (New York, N.Y.).

See also: Salem and Zygmund, p. 891; Dvoretzky, p. 946; Hodges and Lehmann, p. 947; Aitchison and Brown, p. 957; Blackwell, p. 979; Kometani and Kato, p. 979; Rozenblat-Rot, p. 980.

STATISTICS

★**Berkson, Joseph. Estimation by least squares and by maximum likelihood.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. I, pp. 1–11. University of California Press, Berkeley and Los Angeles, 1956. $6.00.

This paper presents small-sample results obtained by experimental sampling on the variance and bias of estimates of the two parameters in the cumulative normal function and the logistic function for describing mortality as a function of dosage. Six different procedures are compared for estimating both parameters simultaneously, and for estimating each parameter separately for a given value of the other. The six procedure studies are: Fisher's maximum likelihood; Neyman's minimum χ^2; Neyman's minimum reduced χ^2; and Berkson's minimum transform χ^2; and two least squares methods. With respect to the first four methods the results indicate that the smallest variances usually occur for the minimum transform χ^2, the next smallest for the minimum χ^2 method, next for the maximum likelihood, and the largest for the minimum reduced χ^2 method. One of the least squares methods compared favorably with the minimum transform χ^2 method, while the other gives variances about like the maximum likelihood method. *S. S. Wilks.*

★**Birnbaum, Z. W. On a use of the Mann-Whitney statistic.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. I, pp. 13–17. University of California Press, Berkeley and Los Angeles, 1956. $6.00.

Let X and Y be independent random variables with continuous cumulative probability functions, and let $X_1, X_2, \cdots, X_m$; $Y_1, Y_2, \cdots, Y_n$ be samples of X and of Y. Put $p=\Pr\{Y<X\}$. The author considers the following problem: to obtain a statistic ψ and, for any $\varepsilon, \alpha>0$, a pair of numbers $M_{\varepsilon,\alpha}$, $N_{\varepsilon,\alpha}$ so that $\Pr\{p\leq\psi+\varepsilon\}\geq1-\alpha$, if $m\geq M_{\varepsilon,\alpha}$, $n\geq N_{\varepsilon,\alpha}$. The possible solution suggested by Lehmann's theorem [Ann. Math. Statist. **22** (1951), 165–179; MR **12**, 726] on the Mann-Whitney statistic is inapplicable in many practical situations because the hypotheses of the theorem are not all fulfilled. As a possible choice for ψ the author suggests a statistic p_1 with the property that $p-p_1\leq D_n^+$, where D_n^+ is the one-sided Smirnov statistic. This choice in conjunction with the tables of Birnbaum and Tingey [ibid. **22** (1951), 592–

596; MR 13, 367] yields a solution to the proposed problem. *R. Pinkham.*

★ **Chernoff, Herman; and Rubin, Herman. The estimation of the location of a discontinuity in density.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. I, pp. 19–37. University of California Press, Berkeley and Los Angeles, 1956. $6.00.

Suppose x is a random variable having the following density function on [0, 1]:

$$f(x)=\begin{cases}\beta, & 0\leq x\leq\alpha,\\ \gamma, & \alpha<x\leq 1,\end{cases}$$

where $\alpha\beta+(1-\alpha)\gamma=1$ and $\gamma<\beta$. This paper is concerned with the asymptotic distribution of the maximum likelihood $\hat{\alpha}$ of α (the location of the discontinuity in $f(x)$) in a sample of size n from this distribution. The authors show that the limiting distribution of $n(\hat{\alpha}-\alpha)$ as $n\to\infty$ is equivalent to the distribution of a random variable A_r where R is the value of r which minimizes B_r where

$$B_r=\begin{cases}y_1+\cdots+y_r-r\dfrac{\log\beta-\log\gamma}{\beta-\gamma}, & r>0\\ (r+1)\dfrac{\log\beta-\log\gamma}{\beta-\gamma}-(y_{r+1}+\cdots+y_0), & r<0,\end{cases}$$

and where

$$A_r=\begin{cases}y_1+\cdots+y_r, & r>0,\\ -(y_{(r+1)}+\cdots+y_0), & r<0,\end{cases}$$

the y's being (non-negative) independent random variables, such any y with a positive subscript has the exponential density function $\gamma e^{-\gamma y}$ and any y with a negative subscript or 0 has the exponential density function $\beta e^{-\beta y}$. The limiting distribution of $n(\hat{\alpha}-\alpha)$ is thus reduced to a random walk problem which is then studied and the results applied to the problem. *S. S. Wilks.*

★ **Dvoretzky, Aryeh. On stochastic approximation.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. I, pp. 39–55. University of California Press, Berkeley and Los Angeles, 1956. $6.00.

This paper gives a generalization of the theory of stochastic approximation procedures motivated by such cases as the Robbins-Monro stochastic process for approximating the point at which a regression function takes on a specified value, and the Kiefer-Wolfowitz stochastic procedure for approximating the point where the maximum of a regression function occurs. The basic theorem of the paper may be stated as follows: suppose α_n, β_n, γ_n $(n=1, 2, \cdots)$ are sequences of non-negative real numbers such that $\sum_1^\infty\beta_n$ is finite ,$\sum_1^n\gamma_n$ is infinite and $a_n\to 0$ as $n\to\infty$. Let X_n and Y_n $(n=1, 2, \cdots)$ be random variables such that

$$X_{n+1}=T_n(X_1, \cdots, X_n)+Y_n$$

where, for some real θ and for any point $(r_1, \cdots, r_n)$ in euclidean R_n, $T_n(r_1, \cdots, r_n)$ $(n=1, 2, \cdots)$ satisfies

$$|T_n(r_1, \cdots, r_n)-\theta|\leq\max[\alpha_n, (1+\beta_n)|r_n-\theta|-\gamma_n].$$

Then if $E\{X_1^2\}$ and $\sum_{n=1}^\infty E\{Y_n^2\}$ are finite and if the conditional expectation $E\{Y_n|x_1, \cdots, x_n\}$ is 0 with probability 1 for all n, then $\lim_{n\to\infty}E\{(X_n-\theta)^2\}=0$ and $P\{\lim_{n\to\infty} X_n=\theta\}=1$.

This result is further generalized by replacing α_n, β_n and γ_n by non-negative functions of points in R_n and satisfying certain regularity conditions.

The author considers several special cases including the Robbins-Monro process and the Kiefer-Wolfowitz process, showing how the specific results previously obtained can be improved. *S. S. Wilks* (Princeton, N.J.).

★ **Ehrenfeld, Sylvain. Complete class theorems in experimental design.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. I, pp. 57–67. University of fornia Press, Berkeley and Los Angeles, 1956. $6.00.

Suppose $Y(x_\alpha)$ $(\alpha=1, \cdots, N)$ are uncorrelated random variables with common variance σ^2 and $x_1, \cdots, x_N$ are N points in a region A in euclidean R_k, such that $E(Y(x)_\alpha)=\theta$, $x_{\alpha 1}+\cdots+x_{\alpha k}$, $(\theta_1, \cdots, \theta_k)$ being an unknown parameter point in a parameter space Ω. This paper is an abstract treatment of the problem of optimum choice of the x's in A for the least squares estimation of linear functions of the parameters, say $t_1\theta_1+\cdots+t_k\theta_k$, where $(t_1, \cdots, t_k)$ is any point in some region T in euclidean R_k. *S. S. Wilks* (Princeton, N.J.).

★ **Elfving, G. Selection of nonrepeatable observations for estimation.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. I, pp. 69–75. University of California Press, Berkeley and Los Angeles, 1956. $6.00.

Suppose $x_1, \cdots, X_N$ are observations of form $x_i=u_{i1}\alpha_1+\cdots+u_{ik}\alpha_k+\xi_i$ $(i=1, \cdots, N)$, where the u's are known, the α's are unknown and the ξ's are uncorrelated random variables with θ means and unit variances. The problem considered in this paper is that of selecting a subset of $n(<N)$ of the observations for estimating the linear form $c_1\alpha_1+\cdots+c_k\alpha_k$ (c's given) with minimum variance. The solution proposed by the author can be stated as follows: For each x_i there is a w_i defined as $\sum_{t=1}^k g_t u_{it}$, where the g_t $(t=1, \cdots, k)$ satisfy the equations $\sum_{t=1}^k \lambda_{st}g_t=c_s$ $(s=1, \cdots, k)$ and where $\lambda_{st}=N^{\frac{1}{2}}\sum_{i=1}^N u_{is}u_{it}$. The n x_i to be chosen are those corresponding to the n largest values of $|w_i|$. The argument for this solution is based on a model in which the N points $u_i, \cdots, u_{ik})$ $(i=1, \cdots, N)$ in k-space are "smoothed" by a continuous density function $f(u_1, \cdots, u_k)$ in k-space, such that f is constant on each member of a family of homothetic ellipsoids. (An example of such a function is the k-dimensional normal distribution.) The author also considers the continuous analogue of the selection problem for the case where f is only centrally symmetric. *S. S. Wilks* (Princeton, N.J.).

★ **Grenander, Ulf; and Rosenblatt, Murray. Some problems in estimating the spectrum of a time series.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. I, pp. 77–93. University of California Press, Berkeley and Los Angeles, 1956. $6.00.

This paper is essentially a review of the probability theory, together with some discussions of the applications, of the stochastic process $y_t=x_t+m_t$, where $E(y_t)=m_t$ and $E(x_t)=0$, x_t being a stationary process. The discussion covers moving averages and linear processes, spectral representation of the process y_t, estimation of the spectral density of the process (including the case where t is multidimensional), and estimation of the cospectrum and quadrature spectrum. *S. S. Wilks.*

★ **Hodges, J. L., Jr.; and Lehmann, E. L. Two approximations to the Robbins-Monro process.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. I, pp. 95–104. University of California Press, Berkeley and Los Angeles, 1956. $6.00.

The Robbins-Monro process is a stochastic process for approximating the value of x at which a regression function $M(x)$ takes a specific value α. More precisely, suppose $Y(x)$ is a random variable for each value of x such that $E(Y(x))=M(x)$. If $M(x)=\alpha$ has a single root θ such that $(x-\theta)$ $(M(x)-\alpha)>0$ for $x\neq\theta$, then for an initial value of x say x_1, and a sequence of positive numbers $\{a_n\}$ the $(n+1)$st approximation to θ is defined by

$$x_{n+1}=x_n-a_n[Y(x_n)-\alpha].$$

Robbins and Monro, as well as Wolfowitz, have studied conditions under which the sequence of random variables $\{X_n\}$, tends to θ in mean square. Blum and Killianpur have investigated conditions under which the sequence converges to θ with probability 1. Chung has concerned himself with the choice of the a_n to speed up the convergence of $\{X_n\}$. He considers two choices of a_n, namely, $a_n=c/n$ and $a_n=1/n^{1-\epsilon}$ for some $\epsilon>0$. For these two choices of the a_n Chung showed that the distributions of $n^{\frac{1}{2}}(X_n-\theta)$ and $(n^{1-\epsilon})^{\frac{1}{2}}(X_n-\theta)$, respectively, converged to normal distributions with θ mean and constant variances.

For the first choice, Chung established certain optimum properties of the estimators X_n, but using the assumption that $M(x)/x\to 0$ as $|x|\to\infty$.

In the present paper the authors show that this last assumption can be removed and that other assumptions of Chung can be slightly weakened. They also investigate the Robbins-Monro process one obtains by approximating $M(x)$ by a linear function of x near θ, in which case they are able to determine the exact variances of the successive estimates X_n of θ and compare them with the variance of the limiting distribution of X_n for the exact $M(x)$.

S. S. Wilks (Princeton, N.J.).

★ **Hoeffding, Wassily. The role of assumptions in statistical decisions.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. I, pp. 105–114. University of California Press, Berkeley and Los Angeles, 1956. $6.00.

This paper is concerned with problems expressed in their most general form in the following way. Suppose F is the distribution function of a random variable (which may be multi-dimensional) under study. Consider two classes of distribution functions $\mathscr{C}_1$ and $\mathscr{C}_2$. Let H_1 be the hypotheses that $F\in\mathscr{C}$, and H_2 the alternative hypothesis $F\in\mathscr{C}_2$. For any given decision rule d in some admissible set of decisions D a risk function $r(F, d)$ defined for all $F\in\mathscr{C}_1\cup\mathscr{C}_2$ and $d\in D$ is assumed given. Suppose d_i is a decision rule which is optimal in some sense under H_i $(i=1, 2)$. Then if d_i $(i=1, 2)$ is unique except for equivalence in $\mathscr{C}_1\cup\mathscr{C}_2$ (i.e. if d_i is also optimal under H_i $(i=1, 2)$, then $r(F, d_i')=r(F, d_i)$ for all $F\in\mathscr{C}_1\cup\mathscr{C}_2$), the consequences of stating that H_1 is true when in fact H_2 is true, can be examined by comparing $r(F, d_1)$ and $r(F, d_2)$ under H_2. If the optimal rules are not unique, a subclass of the optimal rules for H_1 can be chosen which come closest to optimality under H_2 and their performance compared with that of rules which are optimal under H_2.

Various special situations are examined in the framework of this general approach. Included among these is a study of the consequences of assuming that a random variable X has a density function as against the assumption that X is a multiple of some number h. Another is a study of the extent to which certain decision rules derived under the assumption of normality retain their optimal properties when the normality assumptions are relaxed. A third is a study of criteria for the distinguishability of two classes of distributions by means of a randomized test. A fourth is a similar study by means of a sequential test. *S. S. Wilks* (Princeton, N.J.).

★ **Karlin, Samuel. Decision theory for Pólya type distributions. Case of two actions, I.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. I, pp. 115–128. University of California Press, Berkeley and Los Angeles, 1956. $6.00.

Karlin and Rubin [Ann. Math. Statisi. **27** (1956), 272–299; MR **18**, 425] have studied the form of essentially complete strategies in the case of monotone likelihood ratios and have determined the form of Bayes and complete strategies. In the case of testing one alternative hypothesis against another, they assume the difference of the loss functions to have only one sign change (in the parameter space). In the present paper Karlin relaxes this assumption about loss functions, but he deals with more specialized distribution functions, namely, exponential and Pólya type distribution functions. The form of complete strategies, Bayes strategies and admissibility conditions are determined. *S. S. Wilks* (Princeton, N.J.).

★ **Le Cam, L. On the asymptotic theory of estimation and testing hypotheses.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. I, pp. 129–156. University of California Press, Berkeley and Los Angeles, 1956. $6.00.

This paper presents a body of general theory designed to reduce problems of estimation and problems of testing statistical hypotheses to similar problems based on asymptotically normal distributions. The methods developed by the author yield results similar to those presented by Wald [Trans. Amer. Math. Soc. **54** (1943), 426–482; MR **7**, 20] with a somewhat wider choice of freedom of estimates. The methods provide a bridge between maximum likelihood estimates and Neyman's best asymptotically normal estimates [Proc. Berkeley Symposium on Math. Statist. and Probability, 1945, 1946, Univ. of California Press, 1949, pp. 239–273; MR **10**, 388], and also encompass some of Neyman's recent results on certain families of asymptotic tests of composite statistical hypotheses [Trabajos Estadist. **5** (1954), 161–168; MR **16**, 729]. The author also shows how to obtain asymptotically normal and asymptotically sufficient estimates by a process of averaging logarithmic derivatives of distribution of densities, even though maximum likelihood estimates may not be consistent or may not even exist.

S. S. Wilks (Princeton, N.J.).

★ **Robbins, Herbert. An empirical Bayes approach to statistics.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. I, pp. 157–163. University of Cali-California Press, Berkeley and Los Angeles, 1956. $6.00.

Let X be a random variable having discrete values x

and a probability distribution depending in a known way on an unknown real parameter Λ,

$$p(x|\lambda)=P[X=x|\Lambda=\lambda],$$

and Λ itself is considered to be a random variable with a priori distribution function

$$G(\lambda)=P[\Lambda\leq\lambda].$$

We wish to estimate Λ with an estimator of the form $\varphi(X)$. The best such estimator in the sense of minimizing

$$E[\varphi(X)-\Lambda]^2=\sum_x\int p(x|\lambda)[\varphi(x)-\lambda]^2dG(\lambda)$$

is given by $\varphi_G(X)$ where

$$\varphi_G(x)=\frac{\int p(x|\lambda)\lambda dG(\lambda)}{\int p(x|\lambda)dG(\lambda)}.$$

In general $G(\lambda)$ is unknown; but if from available information one can construct an approximation G^* to G, then one would naturally examine

$$\varphi_{G^*}(X)=\frac{\int p(X|\lambda)dG^*(\lambda)}{\int p(X|\lambda)dG^*(\lambda)}$$

as an estimator for Λ.

Consider the case of the experimenter who has independent samples of the form

$$(X_1, \Lambda_1), (X_2, \Lambda_2), \cdots, (X_n, \Lambda_n),$$

where $p(\cdot|\cdot)$ is fixed and known and $G(\cdot)$ is fixed and unknown. Suppose $X_1, \cdots, X_n$ to be known to the experimenter but $\Lambda_1, \cdots, \Lambda_n$ unknown. He wishes to estimate Λ_n from X_n.

Let the unconditional probability distribution of X be given by

$$p_G(x)=\int p(x|\lambda)dG(\lambda).$$

One has a natural approximation to p_G the empirical frequency

$$p_n(x)=1/n\{\text{number of terms } X_1, \cdots, X_n \text{ which equal } x\}.$$

The problem then is to obtain from $p_n(x)$ an approximation to the unknown G, or at least, in the present case, to the functional φ_G. The author considers four classes of distributions to which $p(\cdot|\cdot)$ might be assumed to belong, Poisson, geometric, binomial, and "Laplacian". For example in the Poisson case

$$\varphi_G(x)=(x+1)\,\frac{p_G(x+1)}{p_G(x)}$$

a natural approximation is

$$\varphi_n(x)=(x+1)\,\frac{p_n(x+1)}{p_n(x)}$$

This suggests using $\varphi_n(X_n)$ as an estimator of Λ_n. There are similar results in the other three cases. The author does not examine the properties of these estimates since his purpose is more to point up ideas.

The paper closes with a discussion of estimating G from an approximation to F_G where

$$F_G(x)=\int F(x|\lambda)dG(\lambda)$$

and $F(x|\lambda)$ is known. *R. Pinkham.*

★ **Rosenblatt, Murray. Some regression problems in time series analysis.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. I, pp. 165–186. University of California Press, Berkeley and Los Angeles, 1956. $6.00.

Consider a (vector-valued) time series

$$y_t=\beta_1\phi_{1t}+\cdots+\beta_s\phi_{st}+x_t,$$

$(t=\cdots, -1, 0, +1, \cdots)$, where x_t is a weakly stationary process, $\phi_{1t}, \cdots, \phi_{st}$ are known, and $\beta_1, \cdots, \beta_s$ are unknown regression coefficients. This paper is concerned with least-square and Markov estimates of the β's. Results are obtained on the asymptotic behavior of the covariance matrices of these two systems of estimates. The author also discusses the conditions under which the least squares estimates are asymptotically as good as the Markov estimates. *S. S. Wilks* (Princeton, N.J.).

★ **Stein, Charles. Efficient nonparametric testing and estimation.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. I, pp. 187–195. University of California Press, Berkeley and Los Angeles, 1956. $6.00.

The author discusses an idea for problems of efficient testing of hypotheses and estimation in the non-parametric case by essentially reducing the problem to a set of problems with given distributions. If the proposed non-parametric procedure does as well (asymptotically for large samples) for each problem in the set as any procedure for that problem, then he considers the non-parametric procedure efficient for large samples. He illustrates the idea with three examples. The first is testing the hypothesis that the median ξ of a symmetric density function is 0 against the alternative that it is positive. The second is in estimating the difference between location parameters and the ratio of scale parameters of two density functions differing only by location and scale. The third is concerned with estimation of a linear relation. *S. S. Wilks* (Princeton, N.J.).

★ **Stein, Charles. Inadmissibility of the usual estimator for the mean of a multivariate normal distribution.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. I, pp. 197–206. University of California Press, Berkeley and Los Angeles, 1956. $6.00.

In this paper the author shows that if $X_1, \cdots, X_n$ are independent random variables having unknown means $\xi_1, \cdots, \xi_n$ and common variance 1, then for $n\geq 3$ $X_1, \cdots, X_n$ is an inadmissible estimator for the vector $\xi_1, \cdots, \xi_n$ (i.e. is not the estimator with the smallest mean square error). He shows that for sufficiently large a and small b $(1-b(a+X_i^2)^{-1})X_i$ $(i=1, \cdots, n)$ has a smaller mean square error for estimating $\xi_1, \cdots, \xi_n$. He also shows that if a particular spherically symmetric estimator for $\xi_1, \cdots, \xi_n$ is admissible with reference to the class of all spherically symmetric estimators, it is admissible with respect to all estimators. He obtains an upper bound for the possible improvement of a spherically symmetric estimator over the usual estimator $X_1, \cdots, X_n$. *S. S. Wilks.*

★ **van der Waerden, B. L. The computation of the X-distribution.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. I, pp. 207–208. University of California Press, Berkeley and Los Angeles, 1956. $6.00.

The author has in a previous article [Math. Ann. 126

(1953), 93–107; MR **15**, 46] defined a statistic X for testing whether two samples have been taken from the same distribution. The large sample asymptotic normal theory gives a poor approximation for small samples, and the author here gives a new approximation for the distribution function of X from which one may compute desired percentage points for small samples.

R. Pinkham.

★**McVittie, G. C. Galaxies, statistics and relativity.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. III, pp. 69–74. University of California Press, Berkeley and Los Angeles, 1956. $6.25.

Summary, without proofs or details, of the results of an earlier paper Astr. J. **60** (1955), 105–115 [MR **16**, 1163].

J. W. Tukey.

★**Neyman, Jerzy; and Scott, Elizabeth L. Statistics of images of galaxies with particular reference to clustering.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. III, pp. 75–111. University of California Press, Berkeley and Los Angeles, 1956. $6.25.

Review of the authors' recent work in this field [cf., e.g., MR **14**, 803; **15**, 357; **16**, 869; **17**, 419, 420]. New results in two directions, (i) models of multiple clustering of galaxies, and (ii) characteristics of images of clusters, are mentioned without detail. These include (for i) the generating function for numbers of images in two solid angles and (for ii) an expression for the distribution of the apparent radius of a visible cluster. *J. W. Tukey.*

★**Blanc-Lapierre, André; and Tortrat, Albert.** Proc. Third Berkeley Symposium on Mathematical Statistics and Probability (1954–55), vol. III, pp. 145–170. University of California Press, Berkeley and Los Angeles, 1956. $6.25.

This paper concerns aspects of statistical mechanics related to the large number of degrees of freedom (rather than ergodic theory); specifically, it concerns construction of asymptotic formulas as $n\to\infty$. Assuming that the time-average of a phase-function can be replaced by its phase-average (either from metric indecomposability of the Hamiltonian system S, or from restriction to "ergodic functions"), the main problem becomes the calculation of certain volumes in the phase-space Γ. This can be reduced to classical problems of the theory of conditional probability; but the reduction "results more from the similarity of the expressions than from the nature itself of the problems." Let S consist of components S_i, and let S have for Hamiltonian $H(M)=\sum_{i=1}^{n} H_i(M_i)$, $M_i\in\Gamma_i$, $M\in\Gamma$, $\Gamma=\Gamma_1\times\cdots\times\Gamma_n$. Let Σ_E denote the constant energy surface $H=\text{const.}\equiv E$, and let $V(E)$ denote the volume inside Σ_E; the problem is to compute $\Omega(E)\equiv dV/dE$. Introduce a fictitious probability distribution (pr.d.) Π of $M\in\Gamma$ such that the S_i components are independent random variables M_i and $E_i(M)$, having in Γ_i the pr.d. densities $f_i(E_i;\alpha)=e^{-\alpha E_i}/\Phi_i(\alpha)$ and $\gamma_i(E_i;\alpha)=\Omega_i(E_i)f_i$, respectively, where $\Phi_i=\int_{\Gamma_i}e^{-\alpha E_i}dV_i$ and $\alpha>0$ is arbitrary. The conditional pr.d. (c.pr.d.) of this ("canonical distribution") Π relative to a fixed E concides with the micro-canonical pr.d. Π_E on Σ_E; indeed, only Π_E has any physical meaning. It is shown (by a modification of Khinchin's proof) that to every $E_0>0$ there corresponds a unique α for which, in Π, the mean $\overline{E(\alpha)}=E_0$. Divide S into $N_c+N_r=n$ components; let $N_r\to\infty$ as $n\to\infty$, and assume "stability of the intensive properties of matter" (Fowler). Then Gibbs' law states: $f(M_c|E_0)$, the c.pr.d. for a "small component" (i.e. N_c bounded), is equal to the a priori pr.d. given by Π, provided α is chosen as above. A short heuristic proof of Gibbs' law, due to K. Itô and based on the law of large numbers, is sketched. A second heuristic proof deals with the special case of an ellipsoidal Σ_E; the authors proceed to render this argument rigorous and general by (a) quoting the version of the central limit theorem given in Khinchin, "Mathematical foundations of statistical mechanics" [OGIZ, Moscow-Leningrad, 1943; MR **8**, 187; **10**, 666] and (b) giving a detailed application of it. Finally the authors present their own proof of Gibbs' law, based on characteristic functions (ch.f.'s). Following Bartlett [J. London Math. Soc. **13** (1938), 62–67] a formula for the ch. f. of a c.pr.d. is first derived, and this proves amenable to simple asymptotic evaluation. Similarly the c.pr.d. of the energy E of a large component ($N_c\to\infty$) is proved asymptotically normal, first by Khinchin's method of pr.d. densities, and then by the author's method of ch.f.'s. Granted Fourier's integral theorem, the latter method is briefer and simpler than the former, even though it requires repetition of a primitive "central limit" proof in each application. In parallel with these problems from classical mechanics, quantum-statistical problems concerning the distribution of different components among different possible quantum states (for both Fermi-Dirac and Bose-Einstein statistics) are treated. In several examples previously studied by Fowler, cases of large and small components are dealt with by the ch.f. method. The authors [cf. C. R. Acad. Sci. Paris **237** (1953), 1635–1637; **240** (1955), 2115–2117; MR **15**, 491; **16**, 1189] thus appear to be the first to show precisely how, by systematic application of present-day probability theory techniques, Fowler's "often artificial tools" (steepest descents) can be avoided not only in classical but in quantum-statistical mechanics as well. *R. W. Bass.*

★**Kampé de Fériet, J. Random solutions of partial differential equations.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. III, pp. 199–208. University of California Press, Berkeley and Los Angeles, 1956. $6.25.

Continuing his investigations of the solutions of linear partial differential equations with randomly determined boundary values [C. R. Acad. Sci. Paris **237** (1953), 1632–1634; **240** (1955), 710–712; Mémoires sur la mécanique des fluides offerts à M. Dimitri P. Riabouchinsky, Publ. Sci. Tech. Ministère de l'Air, Paris, 1954, pp. 153–169; MR **15**, 449; **16**, 930, 268], the author considers, in particular, harmonic functions in the unit circle and solutions of the heat equation in an infinite rod.

H. F. Trotter.

★**Wiener, Norbert. Nonlinear prediction and dynamics.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. III, pp. 247–252. University of California Press, Berkeley and Los Angeles, 1956. $6.25.

General considerations about prediction theories and their relation to dynamical assumptions. Remark that, in the presence of an invariant measure, exact knowledge of one time history stretching back to $t=-\infty$ and knowledge that some dynamical equations hold suffice to determine the equation. *J. W. Tukey.*

★ **Crow, James; and Kimura, Motoo. Some genetic problems in natural populations.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. IV, pp. 1–22. University of California Press, Berkeley and Los Angeles, 1956. $5.75.

This is a review article dealing with the mathematical theory of genetic changes in natural populations. The first part of the paper was prepared by the first author (Crow) and discusses deterministic models of certain processes of genetic change from generation to generation in large natural populations due mainly to J. B. S. Haldane, R. A. Fisher and Sewall Wright. Only brief reference is made to Haldane's work on the effect of natural selection on sex-linked, dominant, recessive, autosomal and various other types of genes. The author merely states Wright's general formula for the rate of change of the frequency of a specified gene A_i in terms of mutation rates of alleles to A_i, an immigration coefficient, the gene frequency of A_i among immigrants and the genotypic selective value. Considerable discussion is devoted to the general equation for the rate of change in the average phenotypic measure of a population and several special cases including Fisher's fundamental theorem of natural selection (the rate of increase in fitness of any organism at any time is equal to its genetic variance at that time). Some attention is given to the modification of this equation to take account of the effects of mutation and migration.

The second part of the paper was written by the second author (Kimura) who treats the change in gene frequency over long periods of time as a continuous stochastic process. In the simplest case of one gene locus and two alleles A_1 and A_2, the probability density function $\phi(x, t)$ of the frequency x of one of the alleles, say A_1, after t generations, satisfies (as originally shown by Sewall Wright) a partial differential equation which is equivalent to the Fokker-Planck equation in physics. An extension of this equation to the case of several loci and several alleles at each loci is written down. It turns out to be an application of Kolmogorov's forward fundamental differential equation giving the law of forward progression of the state of gene frequencies. Considerable attention is given to the special case of the partial differential equation satisfied by $\phi(x, t)$ when random sampling of gametes due to random mating within generations is the only source of random fluctuation. The resulting "random drift" effect for a pair of alleles (first studied mathematically by R. A. Fisher) is presented in detail. The second author's own published results on "random drift" due solely to gamete sampling for the cases of three or more alleles are reviewed in detail, asymptotic solutions being given for a large numbers of generations. Kimura finally reviews, for the case of a pair of alleles (with no dominance) in a large population where effects of gamete sampling are negligible, the random fluctuation in ϕ due to the selective advantage of one allele, say A_1, over the other, and also the random fluctuation in ϕ due to random variations in immigration rate from generation to generation.

S. S. Wilks (Princeton, N.J.).

★ **Dempster, Everett R. Some genetic problems in controlled populations.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. IV, pp. 23–40. University of California Press, Berkeley and Los Angeles, 1956. $5.75.

This is a review article on new analysis of variance methods of attacking problems of non-additive genetic variation arising in populations subjected to prolonged selection or inbreeding and subsequent crossing which have been recently developed by V. L. Anderson and O. Kempthorne [Genetics **39** (1954), 883–898], C. C. Cockerham [ibid. **39** (1954), 859–882], B. I. Hayman and K. Mather [Biometrics **11** (1955), 69–82], and others.

The author presents in some detail the Anderson-Kempthorne procedure for analysis of genetic variation for populations produced by successive self-fertilizations of an F_1 population obtained by crossing two homogeneous parents. He notes that Anderson and Kempthorne have adapted their procedure so as to utilize data from parental lines, backcrosses, selfed backcrosses, and from other genetic selection processes. The Hauman-Mather method of partitioning epistatic variance in the case of inbred and backcrossed populations is discussed briefly with reference to several examples including the case of a dihydrid F_2 population, and is compared with Cockerham's methods for handling similar problems.

S. S. Wilks (Princeton, N.J.).

★ **Neyman, Jerzy; Park, Thomas; and Scott, Elizabeth L. Struggle for existence. The Tribolium model: biological and statistical aspects.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. IV, pp. 41–79. University of California Press, Berkeley and Los Angeles, 1956. $5.75.

This paper presents an account of a joint biological and statistical study of the "struggle for existence" between two species of Tribolium (flour beetles) under certain laboratory conditions. The first part of the paper deals with the biological aspects of the study, the experimental work having been done at the Hull Zoological Laboratory at the University of Chicago. After considerable discussion of the rationale for choosing Tribolium for this type of investigation the authors describe the rather elaborately designed laboratory experiment they used to determine two characteristics of these two species. The first was the steady state population density (per gram of flour) which each species achieved in the vial without the presence of the other under six different temperature-humidity conditions. There were between 60 and 70 replications for each of these conditions. The second was the survival capacity of each species in the presence of the other (the initial populations of the two species being equal) under each of the six temperature-humidity conditions. Under each of the six conditions one of the two species always became extinct (after 25 to 60 generations) while the other continued to flourish. But with the exception of two of the conditions, the winning species was not always the same. In one of the conditions (24°C–30%) one of the species invariably became extinct while in another (29°C–70%) the other species always became extinct.

The second part of the paper is devoted to a consideration of stochastic models for several aspects of the experimental results described in the first part of the paper. The first model is an attempt to explain by a rather simple Markoff process the fact that a single-species population apparently attains an equilibrium. This model is so oversimplified, however, that it leads ultimately either to extinction or to an infinite population, and no modifications are made to make it yield the case of a finite (non-zero) population density.

A second stochastic model is constructed in an attempt to describe the competition between the two species. This

model is also rather over-simplified by assuming: (i) only a passive (egg or pupa) and an active (larva or adult) state of beetle; (ii) that every beetle lays eggs, and (iii) no overlapping of generations. These simplifications together with assumptions about cannibalism (egg-eating) usually made in studies of this kind leads, after some analysis, to formulas for the probabilities $\theta_i(x_n, y_n)$ that a given egg in the vial laid by a beetle of species i $(i=1, 2)$ will survive the required period of time τ_i and become an adult of the $(n+1)$st generation where x_n and y_n are the numbers of beetles of the first and second species in the nth generation. The conditional means of X_{n+1} and Y_{n+1}, the numbers of beetles of the first and second species in the $(n+1)$st generation, for given values of x_n and y_n, are found to be $\bar{\nu}_i x_n \theta_i(x_n, y_n)$ $(i=1, 2)$ where $\bar{\nu}_i$ is the average number of eggs laid per beetle of species i. Formulas for the conditional variances $\sigma_i^2(x_n, y_n)$ $(i=1, 2)$ of X_{n+1} and Y_{n+1} are also given. Assuming $\sigma_{\nu_i}^2<\nu_i$, it follows that $\sigma_i(x_n, y_n)<\sqrt{\bar{\nu}_i x_n \theta_i(x_n, y_n)}$. It is found in this model that the numbers of individuals in the successive generations of species i, in the absence of the other species, tend to fluctuate around the value $\varepsilon_i=\log \bar{\nu}_i/\mu_i\tau_i$ where μ_i is an egg-eating "voracity" constant for species i. In the case of mixed populations, it is shown that for any initial (zero-th generation) mixture of the two species, the statistical interaction of the two species tends to drive the point (X_{n+1}, Y_{n+1}) $(n=1, 2, \cdots)$ into the region between the two parallel lines $\mu_1 x_n+\mu_2 y_n=\mu_i\varepsilon_i$ $(i=1, 2)$ and $\mu_1\varepsilon_1<\mu_2\varepsilon_2$, and thence toward the y-axis, corresponding to extinction of species 2. Considerations are laso given to the case where species i has differential "voracity" constants μ_{ij} for eating the eggs of species j, $i, j=1, 2$.

Further side studies are described including one on the dependence of fecundity on crowing, another on dependence of fecundity on age of beetle, and a study of the distribution sex ratios throughout the various parts of the flour in a vial. *S. S. Wilks* (Princeton, N.J.).

★**Bartlett, M. S. Deterministic and stochastic models for recurrent epidemics.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. IV, pp. 81–109. University of California Press, Berkeley and Los Angeles, 1956. $5.75.

This paper is mainly a review of mathematical theory of recurrent epidemics which has been developed by Hamer, Soper, Wilson, Worcester, the author himself and others. The first model considered is a simple stochastic formulation proposed by Bartlett [J. Roy. Statist. Soc. Ser. B. **11** (1949), 211–229; MR **11**, 672] of the Hamer-Soper model of measles epidemics. If S_t denotes the number of susceptible individuals in a population at time t, I_t the number of infected individuals at time t, then assuming the following transition probabilities during a small time interval δt: $\lambda is\delta t$ for $s\to s-1$, $i\to i+1$, $\mu i\delta t$ for $i\to i-1$ and $\nu\delta t$ for $s\to s+1$ (independently of the previous states of the population); the author shows that for large values of ν/μ and μ/λ a deterministic (nonrandom) solution is obtained in which the moment-generating function of $S_t\lambda/\mu$ and $I_t\mu/\nu$ is approximately $\exp(\theta i_t+\phi s_t)$, with i_t and s_t being number infected and number susceptible at time t on a new scale with equilibrium values of unity, and satisfying equations

$$\frac{di_t}{dt}=\mu i_t(s_t-1), \frac{ds_t}{dt}=\frac{m\mu}{n}(1-i_t s_t).$$

These equations are equivalent to the Soper model of 1929 and are discussed by the author using an approximate solution obtained by neglecting $(s_t-1)(i_t-1)$.

The author also examines $S_t\lambda/\mu-1$ and $I_t\mu/\nu-1$ as stochastic time series and shows that their variances are $\mu^3/\lambda\nu^2+\mu/\nu$ and $\lambda/\mu+\mu/\nu$ respectively, while their covariance is $-\mu/\nu$ approximately, assuming ν/μ and μ/λ large. Several modifications of this model are discussed briefly, including the case of seasonal variability in which the infectivity coefficient λ is replaced by $\lambda+\lambda_1 \cos \omega t$, where the period is one year. Several features of the stochastic model are discussed in contrast to those of the deterministic model. One of these is the instability of the average level in the stochastic model as compared with the equilibrium level in the deterministic model. Formulas are given for the probability of extinction of susceptibles, and for the probability of no epidemic during a time interval t under certain simplifying assumptions.

The author next formulates the problem of epidemics for two population groups by extending the deterministic equations stated above for the case of one group, although he does not give solutions of the equations. He extends the deterministic formulation to many groups, and, in fact, to the case where the infected and susceptibles are continuous density functions of (two) space coordinates and time. Finally, he indicates a stochastic process formulation of this continuous case. *S. S. Wilks.*

★**Bharucha-Reid, A. T. On the stochastic theory of epidemics.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. IV, pp. 111–119. University of California Press, Berkeley and Los Angeles, 1956. $5.75.

This paper discusses possible applications of the theory of age-dependent branching processes, as developed by Bellman and Harris, to epidemics. The Bellman-Harris process, in the language of epidemics, states that: (i) if the incubation period τ for an individual (the length of time he is infected before infecting someone else) is a random variable with distribution $G(\tau)$, if q_n $(n=0, 1, 2, \cdots)$ is the probability he infects n persons, and (ii) if $X(t)$ is a random variable denoting the number of infected persons at time t, then the generating function $\pi(s, t)$ of the probabilities $P(X)t=x)$, defined by $\pi(s, t)=\sum_{x=0}^{\infty} P(X(t)=x)s^x$, satisfies the equation

$$\pi(s,t)=s[1-G(t)]+\int_0^t h[\pi(s,t-\tau]dG(\tau),$$

where $h(s)=\sum_{n=0}^{\infty} q_n s^n$. If the hth moment of $X(t)$ exists, it is obtained by differentiating $\pi(s, t)$ h-times with respect to s and putting $s=1$.

Several spacing cases of the Bellman-Harris process are considered as models for simple epidemic processes. The first is the Galton-Watson deterministic model in which the incubation period is fixed at $\tau=1/\lambda$ and $h(s)=s^2$. It follows that $\pi(s, (k+1)/\lambda)=\pi^2(s, k/\lambda)$, with $\pi(0, s)=s$. Thus, the infected population doubles every time unit of $1/\lambda$. The second is the Yule-Furry simple birth-process in which $h(s)=s^2$, $G(\tau)=1-e^{-\lambda\tau}$ where $\lambda>0$ is the infected rate. In this case the formula for $\pi(s, t)$ is equivalent to the differential equation $\partial\pi/\partial t=\lambda\pi(\pi-1)$ which has been thoroughly discussed by Kendall. The third special case is the simple birth-and-death process where $G(\tau)=1-e^{-\lambda\tau-\mu\tau}$ where μ a "death" rate (i.e. rate at which the infected population decreases). The differential equation in this case is $\partial\pi/\partial t=\lambda\pi^2-(\lambda+\mu)\pi+\mu$. This case has also been

discussed by Kendall, as well as Feller and others. The next special case is one suggested by the author himself in which $G(\tau)$ is the same as in the simple birth-process but has $h(s)=q_0+q_1s+q_2s^2$. It is shown that the basic differential equation in this case is

$$\frac{\partial\pi}{\partial t}=\lambda[q_2\pi^2-(q_0+q_2)\pi+q_0].$$

Solving this differential equation with $\pi(0, s)=s$, and expanding into a power series for s, the author writes out an explicit formula for $P(x|t)=x$) which is A/C for $x=0$, and of form

$$\left(\frac{A}{C}\right)+\left(\frac{B}{D}\right)\left(\frac{D}{C}\right)^x (x=1, 2, \cdots),$$

where

$$A=q_0q_2(e^{\lambda kt}-1),\ B=q_2(q_0-q_2e^{\lambda kt}),\ C=q_2(q_0e^{k\lambda t}-q_2),$$

and $D=q_2^2(e^{\lambda kt}-1)$, with $k=q_0-q_2$. If the epidemic starts with $n>1$ infected individuals at $t=0$, the author shows that $P(X(t)=x)$ is of form

$$\sum_{i=0}^{\infty}\left(\frac{n}{i}\right)\left(\frac{A}{C}\right)^{n-i}\left(\frac{B}{D}\right)^i\left(\frac{-n}{x}\right)\left(-\frac{D}{C}\right)^{x-1}, \text{for } x=n, n+1, \cdots.$$

No expression is given for $P(X(t)=x)$, for $x=1, 2, \cdots n-1$, but for $x=0$, the value is $(A/C)^n$. The remainder of the paper is devoted to a discussion of two problems of hypothesis-testing. The first is that of testing the statistical hypothesis that $\lambda_1<\lambda_2$ where λ_1 and λ_2 are infection rates for two simple birth-processes $X_1(t)$ and $X_2(t)$, using the likelihood ratio as the test criterion. The second is the same problem for two simple birth-and-death processes $X_1(t)$ and $X_2(t)$ assuming the "death" rates μ_1 and μ_2 are equal. *S. S. Wilks.*

★ **Chiang, C. L.; Hodges, J. L., Jr.; and Yerushalmy, J. Statistical problems in medical diagnoses.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. IV, pp. 121–133. University of California Press, Berkeley and Los Angeles, 1956. $5.75.

In this paper the authors confine themselves to a rather general discussion of several statistical problems involved in the statistical evaluation of results of diagnostic aids. A diagnostic aid is a quick and cheap procedure for the early diagnosis of a disease which yields a measurement or measurements on which a decision is made to classify a subject as a suspect or not.

The first problem is that of selcting a critical value of a diagnostic aid variable (measurement) x, say x_0, so as to control, in some sense, the ratio of false positives to true positives in the case of rare diseases. This problem is discussed in terms of a Bayes solution. Thus if π is the proportion of affected individuals in the population, α is the Type I error (probability of a false positive) and γ the Type II error (probability of a false negative) the probability η of a true positive (that a positively diagnosed individual is actually diseased) is

$$\eta=\frac{\pi(1-\gamma)}{\pi(1-\gamma)+(1-\pi)\alpha}.$$

The authors propose that choices of α and γ which make $\eta=\sqrt{\pi}$ might provide a reasonable control of the ratio of false positives to true positives, ignoring such items as cost, mental distress caused in a false positive, public opinion, etc.

The second problem concerns estimation of α and γ in multistage diagnostic aid procedures in which a positive, on the basis of a first diagnostic screening test, is subjected to another diagnostic screening test and so on for several successively more searching tests. In the two-stage case where the positives at the first stage are verified as true or false positives at the second stage the authors find an estimate for α without knowing π. However, it is pointed out that γ cannot be estimated without knowledge of π and without knowledge of which negatives at the first stage are verified as true or false negatives at the second stage.

The rest of the paper is spent in discussing problems of longitudinal diagnosis and its analysis. By longitudinal diagnosis is meant a series of diagnoses taken over a period of time during which significant changes can occur in the progress of a disease. The discussion deals with how information is augmented beyond a series of disconnected diagnoses in such a series and with the problems of establishing normal patterns and significant departures from them. *S. S. Wilks* (Princeton, N.J.).

★ **Cornfield, Jerome. A statistical problem arising from retrospective studies.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. IV, pp. 135–148. University of California Press, Berkeley and Los Angeles, 1956. $5.75.

This paper discusses some confidence interval problems motivated by cancer research studies. The first problem is to determine confidence limits of $p_1q_2/p_2q_1=z$, say, where p_1 and p_2 are parameters in independent binomial distributions, with $q_1=1-p_1$, $q_2=1-p_2$. More specifically, if x_1 and x_2 are independent random variables having binomial distributions

$$\binom{n_1}{x_1}p_1^{x_1}q_1^{n_1-x_1} \text{ and } \binom{n_2}{x_2}p_2^{x_2}q_2^{n_2-x_2},$$

the author considers the conditional distribution for which $x_1+x_2=m$, noting that x_1 has the probability distribution

$$f(x_1)=C\binom{n_1}{x_1}\cdot\binom{n_2}{m-x_1}z^{x_1} (x_1=0, 1, \cdots, n_1),$$

where C is the obvious normalizing factor. He points out that by choosing z_1 so that $\sum_{y=0}^{x'} f(y)=\frac{1}{2}\alpha$ and z_2 so that $\sum_{y=x'}^{n} f(y)=\frac{1}{2}\alpha$, where x' is the observed value of x_1, then (z_1, z_2) is a 100 $(1-\alpha)$% confidence interval for z, although the determination of (z_1, z_2) is any useful case involves heavy calculation. He then considers an approach to the problem in the case of large n_1 and n_2. He shows that if $\bar{x}_1$ is the value of x_1 for which $f(x_1)$ has its maximum value, than for large n_1 and n_2, x_1 has approximately a normal distribution with mean $\bar{x}_1$ and variance

$$\left(\sigma_{\bar{x}_1}{}^2=\frac{1}{\bar{x}_1}+\frac{1}{n_1-\bar{x}_1}+\frac{11}{m-\bar{x}_1}+\frac{1}{n_2-m+\bar{x}_1}\right).$$

An approximate 100 $(1-\alpha)$% confidence interval $(x, \bar{x})$ for $\bar{x}_1$ is obtained as the set of values of $\bar{x}_1$ for which $-y_\alpha<(x_1-\bar{x}_1)/\sigma_{\bar{x}_1}<+y_\alpha$ is chosen so that the probability is $1-\alpha$ of the unit normal random variable falling in $(-y_\alpha, +y_\alpha)$. But $\bar{x}_1$, is a monotonic function of z namely,

$$\bar{x}_1(n_2-m+\bar{x}_1)=z(n_1-\bar{x}_1)(m-\bar{x}_1).$$

Hence a 100 $(1-\alpha)$% confidence interval (z_1, z_2) for z is given by the two values of z corresponding to $\bar{x}_1=\underline{x}$ and $\bar{x}$. The adequacy of the large sample procedure for small samples is examined in one example with fairly good results.

The author makes two extensions of the large sample procedure. The first is to the problem of confidence regions for the parameters

$$z_{ij}=\frac{p_{ij}p_{rs}}{p_{is}/p_{rj}}\quad (i=1,\cdots,r-1;\ j=1,\cdots,s-1)$$

for the case of s independent multinomial distributions

$$\frac{n_j!}{x_{ij}!\cdots x_{rj}!}(p_{ij})^{x_{ij}}\cdots(p_{rj})^{x_{rj}}\quad (j=1,\cdots,s)$$

subject to the conditions

$$x_{ij}+\cdots+x_{rj}=n_j\quad (j=1,\cdots,s).$$

The second extension is to the case of N pairs of independent binomial distributions. In this case the constraints are $x_{1j}+x_{2j}=m_j$ $(j=1,\cdots,N)$, and the additional constraint $x_{11}+\cdots+x_{1N}=r$. The parameters in this case are

$$\theta_j=\frac{z_j}{z_n}\quad (j=1,\cdots,N-1).$$

The procedures in both of these extensions are straightforward and involve chi-square distributions with $(r-1)(s-1)$ and $(N-1)$ degrees of freedom, respectively. Illustrative examples are given in both cases.

S. S. Wilks (Princeton, N.J.).

★ **Kendall, David G. Deterministic and stochastic epidemics in closed populations.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. IV, pp. 149–165. University of California Press, Berkeley and Los Angeles, 1956. $5.75.

In this paper Kendall reviews deterministic and stochastic theory of epidemics in a closed population. The deterministic model considered is that of Kermack and McKendrick in which x, y, z, the number of susceptibles, infected persons, and removed persons (recovered or dead), satisfy the equations $x+y+z=$constant and $dx/dt=-\beta xy$, $dy/dt=\beta xy-\gamma y$, $dz/dt=\gamma y$, where β and γ are constants. An approximate solution was obtained for these equations by Kermack and McKendrick, for the initial conditions $x=x_0$, $y=y_0$, $z=0$. Kendall, however, obtains the exact solution. He discusses the essential features of the solution and points out the inadequacies of the approximate solution.

In the stochastic analogue of the Kermack-McKendrick deterministic model $X(t)$, $Y(t)$, $Z(t)$ are integer-valued random variables with initial conditions $X(0)=m$, $Y(0)=a$, $Z(0)=0$, such that in time interval $(t, t+dt)$, and except for terms of magnitude $o(dt)$, the probability of infection $((X, Y)\rightarrow(X-1, Y+1))$ is $\beta XYdt$, the probability of a removal $((Y, Z)\rightarrow(Y-1, Z+1))$ is γydt, and the probability of no change is $1-(\beta X+\gamma)Ydt$.

Kendal reports on the numerical results of Bailey's study of the distribution of the number of secondary cases $(X(0)-X(\infty))=(Z(\infty)-a)=w$, say, which ultimately occur in the population according to this stochastic model, and develops a simple approximating stochastic process which yields the following approximation for $E(w)$, where $\rho=\gamma/\beta$:

$$E(w)\cong\frac{ma}{\rho-m},\ m\leq\rho,$$

$$\cong\left(\frac{\rho}{m}\right)^a\cdot\frac{a\rho}{m-\rho}+\left[1-\left(\frac{\rho}{m}\right)^a\right]\cdot(\zeta-a),\ m>\rho,$$

where ζ is the positive root of

$$m+a-\zeta=m\exp(-\zeta/\rho).$$

The final section of the paper presents some numerical results comparing the deterministic and stochastic models for $\beta=1$, $\gamma=10$, $m=20$, $a=1$ over the time interval (0, 1) at increments $\Delta t=.05$. The values of x, y, z for the deterministic model, were obtained by computation from the basic equations, while those for the stochastic model were obtained by averaging $X(t)$, $Y(t)$, $Z(t)$ for 20 Monte Carlo realizations of the stochastic process. The deterministic and stochastic models give quite different results.

For the case $m>\rho$ the approximating stochastic process yields two types of behavior. In the first type, the approximation $\bar{Y}(t)$ for $Y(t)$ behaves like the total number of individuals in a simple birth-and-death process with the birth-rate $m\beta$ and death-rate γ and satisfying $\bar{Y}(0)=a$, $\bar{Y}(\infty)=0$. In the second type $\bar{Y}(t)$ behaves like y in the deterministic model. The probabilities of the first and second types of behavior are $(\rho/m)^a$ and $1-(\rho/m)^a$. Of the 20 Monte Carlo epidemics (with $\rho/m=\frac{1}{2}$, $a=1$), 11 seemed to have the first type of behavior and 9 the second type of behavior — a result reasonably consistent with probabilities of $\frac{1}{2}$ for each type of behavior.

S. S. Wilks (Princeton, N.J.).

★ **Taylor, William F. Problems in contagion.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. IV, pp. 167–179. University of California Press, Berkeley and Los Angeles, 1956. $5.75.

Suppose $P_{m,n}(T_1, T_2)$ is the probability that n accidents will occur to an individual during the time interval (T_1, T_2) given that m accidents occurred during the interval $(0, T_1)$. The present paper gives an account of various conditions under which $P_{m,n}(T_1, T_2)$ has been studied and some of the results. The basic assumptions on $P_{m,n}(T_1, T_2)$ are: (1) that $P_{m,n}(T_1, T_2)\rightarrow P_{m,n}(T_1, T_1)$ as $T_2\rightarrow T_1$ for all m,n and that $P_{m,0}(T_1, T_1)=1$, and $P_{m,n}(T_1, T_1)=0$ $(n\geqq1)$;

$$(2)\quad \lim_{T_2\to T_1}\frac{\partial P_{m,n}(T_1, T_2)}{\partial T_2}\rightarrow\delta_n\phi(m, T_1),$$

where $\delta_n=-1, 1, 0$ for $n=0, 1, \geqq2$. One important case treated is the Polya model where $\phi(m, t)=\lambda(1+\mu m)/(1+\nu t)$ where μ and ν are contagion and time coefficients. The author gives the probability generating function of $P_{m,n}(t_{i-1}, t_i)$ $(n=0, 1, 2)$, from which it follows that n has a negative binomial (Polya) distribution. The solution is extended to yield the joint distribution of the numbers of accidents $n_1, \cdots, n_{s+1}$ in the intervals $(t_0, t_1), \cdots, (t_s, t_{s+1})$ respectively, this distribution being the multidimensional negative binomial (Polya) distribution. A second important case is the Greenwood-Yule-Newbold model for which $\phi(m, t)=\lambda$ where λ is a random variable having the Gamma distribution. In both of these cases it is assumed that accidents are not fatal.

Other situations discussed briefly include: (1) the problem of taking several levels of risk into account simultaneously; (2) modifications required in the Polya and Greenwood-Yule-Newbold models when fatal accidents are taken into account; (3) the probability theory of time intervals between successive accidents; (4) the problem of testing the statistical hypothesis that $\phi(m, t)=\lambda$ against the hypothesis that $\phi(m, t)=\lambda+m\psi$ (where $m=\lambda\mu$) for a given set of times of occurrence of

accidents. The author concludes with discussion of some points concerning epidemic contagion, including some of Bailey's results. *S. S. Wilks* (Princeton, N.J.).

★ **Arrow, Kenneth J.; and Hurwicz, Leonid. Reduction of constrained maxima to saddle-point problems.** Symposium on Mathematical Statistics and Probability, 1954–1955, vol. V, pp. 1–20. University of California Press, Berkeley and Los Angeles, 1956. $5.75.

H. W. Kuhn and A. W. Tucker [Proc. 2nd Berkeley Symposium on Math. Statist. and Probability, 1950, Univ. of California Press, 1951, pp. 481–492; MR 13, 855] proved the following theorem regarding constrained maxima. $f(x)$ is a scalar function of the N-vector x, $g(x)$ is a M-vector function of x, y is a M-vector, and

$$\phi(x, y) = f(x) + y'g(x).$$

$f(x)$ is to be maximized subject to the constraints that each component of x and of $g(x)$ is non-negative. If $f(x)$ and the components of $g(x)$ are concave, and if certain other conditions are satisfied, then there exists a non-negative saddle point of $\phi(x, y)$, and $f(x)$ is maximized there. In the present paper the Kuhn-Tucker theorem is extended by relaxing the concavity assumptions, at the expense of modifying $\phi(x, y)$ and proving the results only locally. *F. J. Anscombe.*

★ **Barankin, Edward W. Toward an objectivistic theory of probability.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. V, pp. 21–52. University of California Press, Berkeley and Los Angeles, 1956. $5.75.

A long summary (without proofs) of the author's conception of probability. The primitive notion for a frequency theory of probability is taken to be, not a series of Bernoulli trials, but a general type of discrete-time nonstationary stochastic process. The author attaches much importance to a redefinition of "utility" as a set function instead of a point function, which leads him to identify utility with probability. He appears to be unaware of the ordinary usage of "utility" in econometrics.

Three quotations will indicate the scope of the paper: (1) "The theory defines no boundaries for itself; it has, quite to the contrary, the force of asserting that when probability is correctly conceived in its intimate connection, nay, identification with other fundamental notions of science, then there emerges the structural unity of all reality, the conceptual oneness of all behavior, whether of physical particles or of machines or of human beings." (2) "Toward the subjectivistic viewpoint [of Ramsey, de Finetti and Savage] we frankly admit our attitude to be one of strong emotion." (3) "The over-all picture now emerges: all reality is one grand stochastic process, and any system is a marginal process of this universal process." *F. J. Anscombe.*

★ **Suppes, Patrick. The role of subjective probability and utility in decision-making.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. V, pp. 61–73. University of California Press, Berkeley and Los Angeles, 1956. $5.75.

An axiomatization of decision theory is given, similar to that in L. J. Savage's "The foundations of statistics" [Wiley, New York, 1954; MR 16, 147]. "The theory presented here differs from Savage's in two important respects: (i) the number of states of nature is arbitrary rather than infinite; (ii) a fifty-fifty randomization of two pure decisions is permitted; this does not presuppose a quantitative theory of probability. ··· The intuitive ideas at its basis were developed ··· in the process of designing experiments to measure subjective probability and utility." A set of eleven axioms for a rational subjective choice structure is proposed. In discussing them, the author distinguishes between structure axioms and rationality axioms. The latter should be satisfied by any rational, reflective man in a decision-making situation; the former impose limitations on the kind of situations to which the theory may be applied. The author shows that his axioms suffice to establish the principle of rational behavior, namely that decision f is weakly preferred to decision g if and only if the expected value of f is at least as great as the expected value of g. *F. J. Anscombe.*

★ **Daniel, Cuthbert. Fractional replication in industrial research.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. V, pp. 87–98. University of California Press, Berkeley and Los Angeles, 1956. $5.75.

Expository paper. *F. J. Anscombe.*

★ **Sobel, Milton. Sequential procedures for selecting the best exponential population.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. V, pp. 99–110. University of California Press, Berkeley and Los Angeles, 1956. $5.75.

We have k populations, such that the lifetimes of components taken from the ith population are distributed in a delayed exponential distribution,

$$\mu_i^{-1} \exp\{-(x-A_i)/\mu_i\}dx \quad (x>A_i),$$

where μ_i and possibly also A_i are unknown. Special interest attaches to the case where all the A_i are known to be equal, and in particular known to be zero. It is desired to select the population having the greatest value for μ_i. Experimental procedures are considered such that the same number n of components are tested simultaneously from each population, and whenever a failure occurs the component is immediately replaced by another from the same population. Various sequential stopping rules are proposed, discussed and compared with a nonsequential nonreplacement procedure. Validation of some of the results is promised in a further paper by R. E. Bechhofer, J. Kiefer and the author. *F. J. Anscombe.*

★ **Anderson, T. W.; and Rubin, Herman. Statistical inference in factor analysis.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. V, pp. 111–150. University of California Press, Berkeley and Los Angeles, 1956. $5.75.

A most thorough exposition of the common-factor model of factor analysis, including recent mathematical results on problems of identifiability, estimation of parameters, testing hypotheses, and determining the number of factors. Some new results are presented, including (a) a proof of the asymptotic normality of the maximum likelihood estimates, (b) equations for obtaining maximum likelihood estimates when certain of the factor loadings are required to be zero. *F. Lord.*

★**Mosteller, Frederick. Stochastic learning models.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. V, pp. 151–167. University of California Press, Berkeley and Los Angeles, 1956. $5.75.

Expository paper on Stochastic learning models for simple psychological experiments, which might serve as an introduction to the book by R. R. Bush and the author [Stochastic models for learning, Wiley, New York, 1955; MR **16**, 1136]. *F. J. Anscombe.*

★**Solomon, Herbert. Probability and statistics in psychometric research: item analysis and classification techniques.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. V, pp. 169–184. University of California Press, Berkeley and Los Angeles, 1956. $5.75.

Mainly an expository paper, including recent work by Rosedith Sitgreaves on the "attenuation paradox." *F. Lord.*

★**Selected papers in statistics and probability by Abraham Wald.** Stanford University Press, Stanford, Calif., 1957. ix+702 pp. (1 plate). $10.00.

A reprinting, by another publisher, of the book reviewed in MR **16**, 435.

Pitman, E. J. G. On the derivatives of a characteristic function at the origin. Ann. Math. Statist. **27** (1956), 1156–1160.

Let $F(x)$ be a distribution with kth moment

$$\mu_k=\int_{-\infty}^{\infty} x^k dF(x)$$

and characteristic function $\phi(t)=\int_{-\infty}^{\infty} e^{itx}dF(x)$. It is known that the existence and finiteness of μ_k is a sufficient condition for the existence and finiteness of $\phi^{(k)}(0)$. Cramér [Mathematical methods of statistics, Princeton, 1946; MR **8**, 39] has shown that when k is even this condition is also necessary, but not when k is odd. Zygmund [Ann. Math. Statist. **18** (1947), 272–276; MR **9**, 88] has given a necessary and sufficient condition for the existence of $\phi'(0)$ by imposing a smoothness condition on $\phi(t)$. In this paper the author imposes conditions on $F(x)$. He shows for odd k that necessary and sufficient conditions for the existence of $\phi^{(k)}(0)$ are (i)

$$\text{(i)} \qquad \lim_{x\to\infty} x^k\{F(-x)+1-F(x)\}=0$$

and (ii) $\lim_{T\to\infty}\int_{-T}^{T} x^k dF(x)$ exists; and, when these two conditions hold, that

$$\phi^{(k)}(0)=i^k \lim_{T\to\infty}\int_{-T}^{T} x^k dF(x).$$

H. P. Edmundson (Santa Monica, Calif.).

Wijsman, Robert A. Random orthogonal transformations and their use in some classical distribution problems in multivariate analysis. Ann. Math. Statist. **28** (1957), 415–423.

The author applies a procedure using orthogonal transformations of random variables to derive the Wishart distribution, the Bartlett decomposition, and the distributions of Hotelling's T^2 and Wilks' generalized variance. In particular, apart from a constant factor, Hotelling's T^2 is obtained as the F variable of the analysis of variance, Wilks' generalized variance as a product of independent χ^2 variables, and the Wishart distribution is simply related to the joint distribution of independent normal and χ^2 variables. [Cf. M. S. Bartlett, Proc. Roy. Soc. Edinburgh **53** (1933), 260–283; G. Elfving, Skand. Aktuarietidskr. **30** (1947), 56–74; MR **9**, 48; A. T. James, Ann. Math. Statist. **25** (1954), 40–75; MR **15**, 726; J. Ogawa, Osaka Math. J. **5** (1953), 13–52; MR **15**, 141.] *S. Kullback* (Washington, D.C.).

Tukey, John W. Variances of variance components. III. Third moments in a balanced single classification. Ann. Math. Statist. **28** (1957), 378–384.

Following the author's earlier work [same Ann. **27** (1956), 722–736; **28** (1957), 43–56; MR **18**, 345, 959] for finding variances of variance components of a sample drawn from a finite population, the techniques are applied to finding the third moment about the mean of an estimate of the between variance component in a balanced single classification. *M. Muller.*

Philipson, Carl. Explicit expressions for the first four moments of a truncated distribution defined by Pearson type VI. Skand. Aktuarietidskr. **39** (1956), 63–69.

Consider the truncated p.d.f. $F(x|0, q)=[F(x)/F(q)$ for $0\leq x<q$; 0 for $x<0$; 1 for $q\leq x]$, where $F(x)=P(Z\lambda_0\leq x)$ and the p.d.f. of Z is an F-distribution with λ_1 and λ_2 degrees of freedom. The author obtains the first four moments about q of the truncated d.f. *I. Olkin.*

Wise, J. Regression analysis of relationships between autocorrelated time series. J. Roy. Statist. Soc. Ser. B. **18** (1956), 240–256.

For recursive systems it is shown that autocorrelation in the residuals of one relation can be incompatible with the assumption that in any relation the residual is uncorrelated with the explanatory variables; if such relations are estimated by least squares regression the autocorrelation may accordingly be accompanied by regression bias. Since Wise brings his point in relief against extensive quotations from the book of the reviewer and Juréen [Demand analysis, Geber, Stockholm, 1952; MR **16**, 274] I take the opportunity to comment: In the system by which Wise illustrates that the bias can be large there is no exogenous variable. The situation seems to be analogous to the case when intercorrelation between residuals of different relations is necessarily accompanied by regression bias [loc. cit., p. 251, Exercises 27–28], in which case the bias will be of small order of magnitude if exogenous variables are present. *H. Wold* (Uppsala).

Lehmann, E. L. A theory of some multiple decision problems. I. Ann. Math. Statist. **28** (1957), 1–25.

By formulating the problem of testing of statistical hypotheses as one involving only two decisions, namely acceptance or rejection of a certain hypothesis, one not only fails to distinguish between the various possible alternatives when the decision is to reject the hypothesis, but one may also be led to an inappropriate region for the acceptance. of the hypothesis. The author describes a general class of multiple decision problems together with procedures that seem appropriate for these problems. The method is shown to be applicable to problems of point estimation and to various nonparametric problems. It is shown that these procedures are unbiased and possess uniformly minimum risk among all procedures that are unbiased with respect to a certain loss function. This provides a justification for several procedures available for certain classes of point estimates and for some nonparametric decision procedures based on sample cumulative functions. *O. P. Aggarwal* (Edmonton, Alta.).

Steinhaus, H. On prognosis. Zastos. Mat. 3 (1956), 1–7. (Polish. Russian and English summaries)

A discussion whose conclusion is that a statistical decision can be optimal only relative to some loss function.
J. Wolfowitz (Ithaca, N.Y.).

Fabian, Václav. Decision functions and the minimax principle. Časopis Pěst. Mat. 81 (1956), 272–286. (Czech)

Expository paper. *J. Wolfowitz* (Ithaca, N.Y.).

Good, I. J. On the estimations of small frequencies in contingency tables. J. Roy. Statist. Soc. Ser. B. 18 (1956), 113–124.

This paper proposes a method of estimating small cell probabilities in large pure contingency tables for which the hypothesis of independence of rows and columns has been rejected. Let x_{rs} be defined by $p_{rs}=p_{r.}p_{.s}x_{rs}$) where the p's are cell, row, and column probabilities respectively. It is postulated that x_{rs} has either a gamma or logarithmico-normal a priori distribution. Then the corresponding expected value of the cell frequency is used in place of the orginal small observed cell frequency in estimating the cell probability p_{rs}, after the parameters of the a priori distribution have been estimated from the observed contingency table. *S. W. Nash* (Vancouver, B.C.).

Blum, J. R.; and Weiss, Lionel. Consistency of certain two-sample tests. Ann. Math. Statist. 28 (1957), 242–246.

Let $X_1\cdots X_m$, $Y_1\cdots Y_n$ denote variates independently distributed on the unit interval, the X's rectangularly and the Y's with prescribed density function. The authors derive an expression for the limiting frequency function $Q(r)$ of the lengths of X runs, and show that the empirical distribution tends to $Q(r)$ with probability one. This result is used to determine the alternative distributions against which the Dixon C^2 test and variants of the Wald-Wolfowitz run test are consistent. An optimality property of the Dixon test is demonstrated. *P. Whittle.*

Tweedie, M. C. K. Some statistical properties of inverse Gaussian distributions. Virginia J. Sci. (N.S.) 7 (1956), 160–165.

In a previous paper [Nature 155 (1945), 453; MR 6, 232] the author was concerned with inverse binomial variates, and now deals with Gaussian variates. Let x be a Gaussian variate with log-characteristic function $\log Ee^{-tx}=aL(t)$, and consider the inverse Gaussian variate y whose log-characteristic function is $bL^{-1}(t)$, a and b are constants. The density function of y is known to be

$$\sqrt{(\tfrac{1}{2}\lambda\pi y^3)}\exp[-\lambda(y-\mu)^2/2\mu^2y]dy\ (0<y<\infty).$$

Estimates of the parameters and their properties are given without proof. The connection with Brownian motion is discussed. *I. Olkin.*

De Munter, Paul. Sur différentes méthodes pour comparer les fonctions de puissance de tests statistiques. Acad. Roy. Belg. Bull. Cl. Sci. (5) 42 (1956), 1159–1177.

This paper surveys a number of related criteria of asymptotic efficiency of tests. These include the one of Pitman, and concern test statistics which are at least asymptotically normally distributed or centrally or noncentrally chi-square distributed. *M. Dwass.*

Chu, J. T. Some uses of quasi-ranges. Ann. Math. Statist. 28 (1957), 173–180.

If $F(x)$ is a c.d.f. for p, $0<p<1$, any ξ_p satisfying

$$F(\xi_p-0)\leqq p\leqq F(\xi_p)$$

is a quantile of order p. In this paper confidence intervals for and tests of hypotheses about $\xi_q-\xi_p$ $(0<p<q<1)$ are given, based on differences of pairs of order statistics, i.e. on differences x_s-x_r where x_r, x_s are the rth and sth ordered observations from a random sample of n from the population with c.d.f. $F(x)$.

This is a simple extension of the procedure to find confidence intervals or tests for ξ_p, using order statistics.
D. G. Chapman (Seattle, Wash.).

Elandt, R. On certain interaction tests in serial experiments. The problem of stratification. Zastos. Mat. 3 (1956), 8–45. (Polish. Russian and English summaries)

The author proposes a statistic U (defined below) for a (non-parametric) test of the hypothesis that the chance variables X, Y are independently distributed, and obtains the small and large sample distribution of U under the null hypothesis. Let (X_i, Y_i) $(i=1, \cdots, N)$ be independently and identically distributed pairs of chance variables with the same distribution as (X, Y), let

$m=\text{median}(x_1, \cdots, X_N)$, $m'=\text{median}(Y_1, \cdot\cdot, Y_N)$, $\xi_1=X_i-m$, $\eta_i=Y_i-m'$. Then U is defined as

$$\sum_{i=1}^{N}\text{sign}(\xi_i\eta_i).$$

J. Wolfowitz (Ithaca, N.Y.).

Bennett, B. M. On a rank-order test for the quality of probability of an event. Skand. Aktuarietidskr. 39 (1956), 11–18.

Consider a sequence of n independent trials in which an event E may occur on the jth trial with probability p_j. The number of occurrences is fixed at k. Random variables x_j, y_j are introduced, where $x_j=j$, $y_j=1$, if E occurs, $x_j=0$, $y_j=0$ if E does not occur on the j-th trial. The author proposes the statistic $X_n=\sum x_j$ to test H_0: $p_j=p$ $(j=1, 2, \cdots, n)$, and determines the conditional m.g.f. given $Y_n=\sum y_j=k$ under H_0 and under a trend alternative H: $\log p_j/(1-p_j)=\tau+\lambda(j-1)$. X_n is an unbiased test and under certain conditions on k/n and λ, is asymptotically normal. *I. Olkin* (East Lansing, Mich.).

Blom, Gunnar. On linear estimates with nearly minimum variance. Ark. Mat. 3 (1957), 365–369.

Let z be a random variable with continuous p.d.f. depending only on location and scale parameters, $z_{(1)}\leqq\cdots\leqq z_{(n)}$ an ordered sample, and $x_i=(z_{(i)}-\mu)/\sigma$. If Ex_i, $\text{cov}(x_i, x_j)$ are known, then μ and σ are estimable by linear MVU estimates based on the ordered observations. The purpose of this paper is to show how one may obtain linear unbiased estimates whose efficiency (in the sense of variance) is near optimal. The method has the advantage of not requiring information about the covariances.
I. Olkin (East Lansing, Mich.).

Taylor, J. Exact linear sequential tests for the mean of a normal distribution. Biometrika 43 (1956), 452–455.

Wald's test for the mean of a normal distribution with known standard deviation involves an approximation which is poor when the alternative hypotheses differ

greatly. The characteristics of some linear sequential tests have been calculated, and it is found that they have the high efficiency customary in sequential probability ratio tests. (From author's summary.) *I. Olkin.*

Gulati, R. L. Sequentielle Tests für den Korrelationskoeffizienten. Mitteilungsbl. Math. Statist. 8 (1956), 202–233.

Let $\{x_i, y_i\}=(x_1, y_1; \cdots; x_i, y_i; \cdots)$ be a sequence of independent sample pairs from a bivariate normal population π with means μ_x, μ_y, standard deviations σ_x, σ_y, and a correlation coefficient ρ. By the transformation $z_i'=[z_{i+1}-(1/i)\sum_{\nu=1}^{i} x_\nu]\{i/(i+1)\}^{\frac{1}{2}}$, earlier applied to univariate samples by, e.g., Irwin [Suppl. J. Roy. Statist. Soc. **1** (1934), 236–251] and Girshick [Ann. Math. Statist. **17** (1946), 123–143; MR **8**, 44], the author first changes $\{x_i, y_i\}$ into a sequence of independent sample pairs $\{x_{i-1}', y_{i-1}'\}$ from a population π' which differs from π only by having zero means. Using $\{x_{i-1}', y_{i-1}'\}$ the author constructs SPR (sequential probability ratio) tests for ρ, using Wald's theory [ibid. **16** (1945), 117–186; MR **7**, 131]. These tests are: directly derived SPR tests, assuming either σ_x, σ_y known or only σ_x/σ_y known; a SPR test based on a "best" linear combination of x' and y', assuming σ_x, σ_y known; a SPR test based on U, defined as 1 if $x'y'\geqq 0$ and as 0 otherwise; and a SPR test based on V, defined as 1 if $(x', y')\in B$ and as 0 otherwise, assuming σ_y, σ_y unknown in the last two cases and taking as B a "best" region of the (x', y')-plane bounded by two straight lines through the origin. *D. M. Sandelius* (Göteborg).

Weiss, Lionel. On the uniqueness of Wald sequential tests. Ann. Math. Statist. 27 (1956), 1178–1181.

Let $\{X_n\}$ be an infinite sequence of independent and identically distributed random variables with densities $f_i(X)$ under the hypotheses H_i $(i=1, 2)$. Let A and B, $B<A$ denote the stopping bounds for the usual Wald sequential probability ratio test, $\alpha(B, A)$ the probability of accepting H_1 when H_2 is true, $\beta(B, A)$ the probability of accepting H_2 when H_1 is true. The property in question is whether a given strength (α, β) for the test uniquely determines (A, B). Under mild restrictions on the distribution of $f_2(X)/f_1(X)$, the author proves that there is at most one solution to the equations $\alpha(B, A)=\alpha$ and $\beta(B, A)=\beta$. *I. Olkin* (East Lansing, Mich.).

★**Aitchison, J.; and Brown, J. A. C. The lognormal distribution, with special reference to its uses in economics.** Cambridge, at the University Press, 1957. xviii+176 pp. $6.50.

A variate is said to be lognormally distributed if its logarithm (or in some cases the logarithm of a translation) is Gaussian. The study of the lognormal distribution dates at least to McAlister in 1879, and it and its nice properties have been rediscovered often since then. One of the many attractive aspects of this book is a bibliography of 217 items, most of which bear directly on the lognormal distribution.

Chapter 2 surveys reproductive and other properties of the distribution, nearly all of which are obvious consequences of the interplay of properties of the logarithmic function and the Gaussian distribution. Thus powers of lognormal variates and products of independent lognormal variates are lognormal, and there is a multivariate generalization to the dependent case. Conversely, the analogue of Cramer's theorem states that if the product of two independent positive variates is lognormal, so is each factor. Multiplicative analogues of the central limit theorem are also given.

It is this property that lends interest to the lognormal distribution in a wide array of applications, since it is easy to formulate random processes which lead to lognormality. There are many versions of the "law of proportionate effect", suggesting that the value of a variable is built up through a sequence of independent random impulses, with the effect of an impulse being proportional to the value already attained. One difficulty is that if this process takes place in time the variance of the resultant random variable ought to increase indefinitely. Various stabilizing devices have been suggested. A slightly different version of the theory has been used by Kolmogoroff and others to explain the occurrence of lognormal distributions of particle sizes in materials crushed naturally or artifically. All this is outlined in Chapter 3.

Chapter 4 describes the construction of 65 artificial samples to be used subsequently as experimental material in the discussion of estimation methods, particularly graphical ones. Chapters 5 and 6 on estimation will be of special interest to statisticians and econometricians. Estimation problems fall naturally into sections depending on whether the observed variate or an unknown translation of it is lognormal, and on whether it is desired to estimate the mean and variance of the observed variate or of its logarithm. The techniques considered are maximum likelihood, moments, the use of quantiles, and graphic estimation using probability paper. Large sample efficiencies are given, as well as the results when all methods are used on the artificial samples. In general quantile estimates turn out to be relatively good and relatively easy. Chapter 9 considers truncated and censored distributions and related problems.

Chapter 7 will contain nothing new for statisticians already acquainted with probit analysis but may provide an interesting dosage for economists. Farrell, Tobin, and the authors have already begun to get interesting results from using probit and related methods in demand analysis, particularly the demand for expensive durable consumer goods, which offers difficulties for standard Engel curve methods.

In Chapters 10–12 the authors turn to applications, particularly in economics. Chapter 10 briefly surveys the empirical occurrence of lognormality in engineering, economics, sociology, biology, demography, philology and elsewhere. Chapter 11 discusses the distribution of incomes by size, the theoretical plausibility and empirical utility of the lognormal hypothesis, and the nature of the resulting Lorenz curve. Chapter 12 is devoted to the rationalization and use of the distribution in Engel curve work. The last chapter briefly describes computational problem. There and 12 pages of tables of functions to facilitate estimation.

This is a well-written and well-made book, bound to be a stimulus and an aid to research workers in econcmics and other fields. *R. Solow* (Cambridge, Mass.).

Standish, Charles. N-dimensional distributions containing a normal component. Ann. Math. Statist. 27 (1956), 1161–1165.

The author obtains necessary and sufficient conditions for an n-dimensional distribution function to have as a factor the distribution function of n independent normal random variables with common mean zero and variance one. More exactly, he obtains conditions for the density

function $f(x_1, \cdots, x_n)$ to be of the form

$$f(x_1, \cdots, x_n)=\pi^{-\frac{1}{2}n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{-[(x_1-u_1)^2+\cdots+(x_n-u_n)^2]\}dP(u_1,\cdots,u_n),$$

where $P(u_1, \cdots, u_n)$ is a distribution function. Here the method of proof for the n-dimensional case differs from the heat equation approach used by Pollard [Proc. Amer. Math. Soc. 4 (1953), 578–582; MR 15, 28] on the one-dimensional case. *H. P. Edmundson.*

Bose, P. K. Normalisation of frequency functions. Bull. Calcutta Math. Soc. 48 (1956), 109–119.

The author determines the error committed in replacing a general sampling distribution by the normal distribution for various sample sizes n. Using Hermite polynomials, the frequency function $f(x)$ and the c.d.f. $F(x)$ of the statistic are expressed in series up to terms of order $O(n^{-1})$. He then calculates the maximum error for the c.d.f. of the statistics t, χ^2, and $(2\chi^2)^{\frac{1}{2}}$ for $n=5(5)25$, $30(10)60$, etc., and of the classical D^2 statistic for $p=1(1)6$ and $\delta=n\Delta^2=200(100)400$, etc. Cornish and Fisher [Rev. Inst. Internat. Statist. 3 (1937), 1] and Bose and Rao [Science and Culture 9 (1943), 402–403; MR 5, 209] have made similar studies. He notes that his results are not very accurate for small values of n.

H. P. Edmundson (Santa Monica, Calif.).

Ito, Koichi. Asymptotic formulae for the distribution of Hotelling's generalized T_0^2 statistic. Ann. Math. Statist. 27 (1956), 1091–1105.

In multivariate analysis of variance the basic probability law is

$$P(X_0, X_1)= \text{const}\exp[-\tfrac{1}{2}\operatorname{tr}\Lambda(X_1-\xi)(X_1'-\xi')-\tfrac{1}{2}\operatorname{tr}\Lambda X_0X_0']\,dX_0dX_1$$

where X_1, ξ are $p\times m$ matrices, X_0, Λ are $p\times p$ matrices. In terms of familiar univariate analysis of variance nomenclature $S_1=(1/m)X_1X_1'$ is the sample "between" dispersion matrix, $(1/m)X_0X_0'$ the corresponding population "between" dispersion matrix, and $S_0=(1/n)X_0X_0'$ is the sample "within" dispersion matrix. Hotelling [Proc. 2nd Berkeley Symposium on Math. Statist. and Probability, 1950, Univ. of California Press, 1951, pp. 23–41; MR 13, 479] proposed a test of the hypothesis H_0: $\xi=0$ based on the statistic

$$T_0^2=m\operatorname{tr}S_1S_0^{-1}.$$

The exact distribution of T_0^2 is not available at present even in the null case $\xi=0$, but it is known that $m\operatorname{tr}S_1\Lambda$ has a χ^2 distribution with mp d.f. Beginning with this the author obtains an expression for percentage points of the T_0^2 distribution as functions of percentage points of the χ^2 distribution, as a series in $1/n$ up to terms of order $O(n^{-3})$. The primary tool used is Taylor series expansion, but in view of the nature of the random variables, the algebraic manipulation is very heavy. The same method also yields an asymptotic formula for the c.d.f. of T_0^2 [given to terms $O(n^{-3})$]. A procedure is given to obtain some information on the bounds of the error in using these asymptotic formulae. *D. G. Ghapman.*

Davies, O. L.; van Dun, F. J.; and Hamaker, H. C. Design and analysis of industrial experiments. Statistica, Neerlandica 9 (1955), 189–207. (Dutch. English summary)

This paper is based on a lecture on the "Design and Analysis of Industrial Experiments" given by Dr. O. L. Davies on the 8th of May 1954 to the Industrial Section of the "Vereniging voor Statistiek". Production, formulation, and testing are distinguished as three separate fields of chemical activity where experimental designs can be applied, and various numerical examples of such experiments are discussed in detail. They consist of a 2^3 factorial design, a 2^4 half replicate and a 2^5 quarter replicate fractional factorial design, and a three-way classification. In a final section the recent designs developed by Box for the exploration of response surfaces are briefly considered. *Author's summary.*

Zindler, Hans-Joachim. Über die Genauigkeit von Streuungsschätzungen durch Gruppensummen. Mitteilungsbl. Math. Statist. 8 (1956), 192–201.

The author discusses an approximation to the usual unbiased estimate of the variance from a large sample which may be obtained by assigning the $n=rk$ sample values at random to r groups of k each and then using the sum of squared deviations of the group means about the sample mean. It appears to the reviewer that the savings in time and effort in using this approximation would be considerable only to an unskilled or a poorly equipped computer. The variance of the estimate obtained in this way is compared with that of the usual estimate for a variety of values of n, r and fourth standard moments of the universe sampled. *C. C. Craig.*

Ghosh, Birendranath. Enumerational errors in surveys. Calcutta Statist. Assoc. Bull. 7 (1957), 50–59.

Expository. *I. R. Savage* (Minneapolis, Minn.).

Hamaker, H. C. Experimental designs in industry. Statistica, Neerlandica 9 (1955), 209–232. (Dutch. English summary)

It is pointed out that in industry owing to the greater speed of modern mass-production we have a greater need for simple experimental designs than in agriculture. The paper makes a special and extensive study of the simplest of all designs: the two-way classification.

A distinction is made between qualitative and quantitative factors; factors for which we do not, and for which we do know the levels. Three cases must then be distinguished: (1) two qualitative factors, (2) one qualitative and one quantitative factor, and (3) two quantitative factors. To these, different methods of variance analysis must be applied; in (1) the customary analysis according to rows and columns is adequate, but in (2) we can effectively use polynomial regression analysis in one direction, and in (3) we can do so in both directions. Examples of each of these types of analysis are provided.

The opinion is expressed that the analysis of variance serves to pick out from among a number of conceivable models, the simplest model that will adequately describe the observations, and further to provide an estimate of the residual fluctuations. The result of an analysis of variance is of no use for technical purposes; the model chosen expressed in the form of a numerical equation is the information to be used in the factory. *Author's summary.*

Tukey, John W. Variances of variance components. II. The unbalanced single classification. Ann. Math. Statist. 28 (1957), 43–56.

Following part I [same Ann. 27 (1956), 722–736; MR 18, 345], a family of weighted estimates for the variances and covariances of the estimates of "between" and "within" variance components of an unbalanced single classification are presented (i.e., subject to certain independence assumptions). The results apply to arbitrary finite populations. Selected numerical examples are given to study the effects that certain sets of weights may have on the variances and covariances of the "between" and "within" variance components.

M. Muller.

★**Grenander, Ulf; and Rosenblatt, Murray. Statistical analysis of stationary time series.** John Wiley & Sons, New York; Almqvist & Wiksell, Stockholm, 1957. 300 pp. $11.00.

This book goes further than any other yet published towards providing a comprehensive and modern account of the subject of stationary time series and their analysis. It covers the more important parts of spectral and prediction theory, describes several technological applications, and examines certain aspects of inference in time series, particularly the nonparametric estimation of the spectral density.

Chapter 1: A general consideration of stochastic processes and their orthogonal representations, characterisation of a stationary process, spectral and moving average representations, particular processes, time and ensemble averages, the effect of linear and non-linear operations on processes (with rectification, rounding-off and intermittent sampling as special cases).

Chapter 2: A detailed account of linear least-square prediction and interpolation, following Kolmogorov's treatment, and fully developing the relation with the Wold decomposition. Estimation of regression coefficients when the residual has a known spectrum.

Much of the material of these two chapters is to be found nowhere else in such a neat and accessible form. Derivations are rigorous, but the lively pace and frequent interspersion of physical examples make for relatively easy reading. There are several interesting comments and results, e.g. the remarks on determinism on p. 78, the interpolation result on p. 85.

Chapter 3 reviews current inferential techniques for finite parameter models: periodogram analysis, variate differences, smoothing, the distribution of correlation coefficients and quadratic forms, Quenouille's and Wold's tests, the fitting of autoregressions. Much recent theory (e.g. the use of ratios of maximised likelihoods) receives no mention. This is a consequence of the authors' opinion, reiterated in justification of the test of Chapter 6, that a finite parameter model will seldom be realistic, so that inferential techniques must be nonparametric to be valid. A consistent protagonist of this viewpoint would of course jettison most current statistical techniques, and abandon attempts at model construction and interpretation. However, a demand for greater sophistication in model-building is not untimely.

Chapter 4 is concerned with estimation of the spectrum, in general by a quadratic function of the observations. Estimates which are linear in the lagged product-sums are called spectrograph estimates and are shown to be asymptotically as good as more general quadratic estimates. The Bartlett, Daniel, Tukey and heterodyne estimates are discussed, and a number of miscellaneous results given. The style from this point on rarely recaptures the trenchancy of the first two chapters, perhaps because the material is so much further from a definitive treatment.

Chapter 5 briefly describes some interesting physical examples of spectra and their measurement: thermal noise, turbulence, ocean waves.

Chapter 6 contains the authors' central result: an asymptotic expression for the distribution of the largest deviation of the sample distribution function from expectation, for the case of a linear process. The calculation differs from Kolmogorov's consideration of a probability distribution in that the ordinates of the density function do not possess the same degree of independence, and are not constrained to integrate to unity. If the variates are normal this limiting distribution depends only upon a single parameter, and is used to construct a confidence band for the spectral density curve. The results are less neat in the case of non-normality, but a conservative confidence region can be found. The method is applied to artificial series. Several miscellaneous questions are treated, such as the application of the methods to residuals from a regression.

Chapter 7 presents the authors' important results on the asymptotic behaviour of the least-square estimates of regression coefficients. The conditions for asymptotic efficiency receive an elegant formulation in terms of the elements of the regression spectrum defined on p. 241. Various polynomial and trigonometric regressions are considered in detail.

Chapter 8 is a miscellany: the effects of prediction of estimation or incorrect specification of the spectrum, zeroes and maxima of processes, tests of normality, and other topics.

The book concludes with a set of 51 problems and an appendix on complex variable theory (complementing Chapter 2), both useful.

The methods proposed in Chapter 6, which motivate the book, are likely to be criticised on the grounds of inexactness, lack of power, or sensitivity to non-normality. Much more serious, however, is the lack of any indication as to how the methods are to be used, and how they are to serve the final purpose of an analysis. Despite the title, not a single series is analysed. The authors seem much more interested in the technological aim of estimating specifications for filter design than in the scientific one of explaining the data.

For this reason much of the original work is likely to be of use to technologists, of interest to statisticians. Taken as a whole, the book is a powerful and stimulating addition to the literature.

P. Whittle (Canberra).

Mittmann, O. M. J. Varianz-Untersuchungen bei stationären Variablen. Arch. Meteorol. Geophys. Bioklimatol. Ser. A. 9 (1956), 519–523 (1957).

Consider a stationary stochastic process with discrete time and form the average of n consecutive observations. The author studies how the variance of this average tends to zero for increasing n. This is used to get an estimate of the variance of the average formed from all the observed values of the process. The reasoning is empirical and based upon some numerical examples. No mention is made of the way in which these things are connected with the spectrum of the process.

U. Grenander.

Francini, Giuseppe. La descrizione statistica delle grandezze elettriche fluttuanti. Ricerca Sci. **26** (1956), 2973–3004.

Expository paper directed to research electrical engineers. *C. C. Craig* (Ann Arbor, Mich.).

Parzen, Emanuel; and Shiren, Norman. Analysis of a general system for the detection of amplitude-modulated noise. J. Math. Phys. **35** (1956), 278–288.

The authors treat a system involving square-law detectors and in which the input $u(t)$ is the sum of a stationary Gaussian noise $y(t)$ modulated with index m by a modulating function $g(t)$, which may or may not be random, and a background stationary Gaussian noise $z(t)$, that is, $u(t)=y(t)[1+mg(t)]+z(t)$. If $E(\overline{W}(T)_n)$ and $E(\overline{W}(T)_{mn})$ are the expected values of the integrator output when the input is respectively unmodulated noise and amplitude modulated noise, then the detection of the presence or absence of signal may be based on the theory of testing the statistical hypothesis that $E(\Delta(T))=0$, where $\Delta(T)=\overline{W}(T)_{mn}-\overline{W}(T)_n$. The general results in terms of the statistics of the noise and modulating function are illustrated by application to the case where the noise spectra are flat and the spectrum of the modulating function and the filter transfer function have Gaussian shapes.

S. Kullback (Washington, D.C.).

See also: des Cloizeaux, p. 899; Pitcher, p. 910; Lotkin, p. 936; Copeland, p. 940; Lukacs, p. 942; Menger, p. 942; Anderson and Goodman, p. 944; Kac, p. 960; Wheelon, p. 969; Matschinski, p. 977.

PHYSICAL APPLICATIONS

Mechanics of Particles and Systems

★ **Левинсон, Л. Е. [Levinson, L. E.] Теоретическая механика с элементами теории механизмов. [Theoretical mechanics and elements of the theory of mechanisms.]** Vses. Uč.-Ped. Izdat. Trud., Moscow, 1955. 447 pp. 9.15 rubles.

Textbook for secondary technical schools.

Linsman, Marcel. Sur la construction d'Euler-Savary en géométre cinématique. Bull. Soc. Roy. Sci. Liège **25** (1956), 366–368.

★ **Меркин, Д. Р. [Merkin, D. R.] Гироскопические системы. [Gyroscopic systems.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 299 pp. 9.45 rubles.

This is a treatise on dynamical systems with special reference to gyroscopic forces, i.e., forces which do no work. The methods employed are generally elementary and the results are for the most part known ones, but there are some novel theorems on stability of linear systems. There are many illustrative examples drawn from mechanics, some concerning gyroscopes. The chapter headings are as follows: 1) Gyroscopic forces, 2) Gyroscopic forces depending on a parameter. 3) Motion of systems subject only to gyroscopic forces. 4) Influence of gyroscopic forces on motion of conservative systems. 5) Influence of gyroscopic forces on motion of nonconservative systems. 5) Stationary motion of gyroscopic systems. There is an appendix on matrices and stability.

W. Kaplan (Ann Arbor, Mich.).

Kondurar', V. T. Decomposition of the force function of mutual attraction of two homogeneous ellipsoids. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki. **10** (1956), 97–107. (Russian)

Leitmann, G. An optimum pursuit problem. J. Franklin Inst. **263** (1957), 499–503.

The optimum thrust direction of a rocket fired from a fighter plane pursuing a constant velocity target is determined with a view towards maximizing the initial stand-off. It is shown that for flight in vacuo the optimum thrust direction remains constant. *Author's summary.*

See also: Wiener, p. 949; Rose, p. 971; Dugas, p. 982.

Statistical Mechanics

Sragovič, V. G. On the probabilistic foundation of the statistics of non-stationary systems. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 768–770. (Russian)

Suppose that, in a system of particles of various types, the particles can combine to produce new types. The author sketches a study of the statistics of this non-stationary system. As one result he states that, under the usual asymptotic conditions, if there are only two types, A, B, and only one combined type AB, the total energies of the particles of types A, B, and the number of type AB, are random variables in an asymptotically normal nondegenerate trivariate distribution for whose parameters he has found simple evaluations.

J. L. Doob (Geneva).

★ **Kac, M. Foundations of kinetic theory.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. III, pp. 171–197. University of California Press, Berkeley and Los Angeles, 1956. $6.25.

This paper is an attempt to justify and derive the Boltzmann equation by the use of the "master equation" [A. Nordsieck, W. E. Lamb, Jr. and G. E. Uhlenbeck, Physica **7** (1940), 344–360; MR **4**, 152; A. J. F. Siegert, Phys. Rev. (2) **76** (1949), 1708–1714] which governs the time rate of change of the N-particle distribution function. The master equation is linear and further, incorporates Boltzmann's Stosszahlansatz. It corresponds to the Kolmogoroff equation [Math. Ann. **104** (1931), 415–458] for the Markoff process taking place on the energy surface.

Let $\Phi_N(\mathbf{R}, 0)$ be a sequence of distribution functions on the sphere $\sum_i x_i^2=N=\|R\|^2$ which have the "Boltzmann property"

$$\lim_{N\to\infty} f_k^N(x_1, \cdots, x_k, 0)=\prod_{j=1}^{k} \lim_{N\to\infty} f_1^N(x_j, 0).$$

The basic theorem is then proved that the solutions $\Phi_N(\mathbf{R}, t)$ propagate the Boltzmann property in time. The non-linear character of the Boltzmann equation from this point of view is thus merely a consequence of the assumed initial conditions.

The connection with the H-theorem is established by demonstrating that $d\int \Phi^2(\mathbf{R}, t)d\sigma/dt\leq 0$ where the integral is over the energy surface. The equality sign only holds for the uniform distribution $\Phi(\mathbf{R}, t)=[S_N(\sqrt{N})]^{-1}$, where $S_N(x)$ is the surface area of the sphere of radius x. Further

$\Phi(\mathbf{R}, t)$ goes to the uniform distribution as $t \to \infty$ in the sense that for every $x(\mathbf{R}) \in L^2(S_N)$,

$$\lim_{t\to\infty} \int \Phi(R,t)\chi(\mathbf{R})d\sigma = \int \chi(\mathbf{R})d\sigma/S_N(\sqrt{N}).$$

The important question of the relation between the master equation and Liouville's theorem is not treated.
E. Frieman.

Albrecht, Rudolf. Zur Darwin-Fowlerschen Methode der statistischen Thermodynamik. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1956**, 205–232 (1957).

This paper presents a new derivation of the Boltzmann, Fermi-Dirac and Bose-Einstein distributions for a single component of a system consisting of infinitely many independent and identical components. The derivation differs from that of Darwin-Fowler in that the author avoids the use of a saddle-point integration. He also claims to have overcome difficulties associated with the assumption that all energy levels are integer multiples of some basic unit of energy although this point may still be open to question. *G. Newell* (Providence, R.I.).

Montroll, Elliott W.; and Shuler, Kurt E. Studies in nonequilibrium rate processes. I. The relaxation of a system of harmonic oscillators. J. Chem. Phys. **26** (1957), 454–464.

Rubin and Shuler [same J. **25** (1956), 59–67, 68–74; MR **18**, 429] derived a set of differential-difference equations to describe the relaxation of a system of harmonic oscillators coupled to a heat bath by means of a perturbation linear in the oscillator coordinates. The equations were similar in type to those which arise in the analysis of the birth and death process in probability theory. An exact solution of these equations is obtained in this paper and used to demonstrate how various types of initial distributions among the energy levels approach the Boltzmann distribution corresponding to the temperature of the bath. One unusual result is the fact that if the system initially has a Boltzmann distribution, it retains a Boltzmann distribution with a time dependent temperature. *G. Newell* (Providence, R.I.).

Guy, Roland. Hydrodynamique en théorie unitaire pentadimensionnelle. Ann. Scuola Norm. Sup. Pisa (3) **10** (1956), 91–117.

L'A. se propose de généraliser en théorie de Jordan-Thiry l'hydrodynamique relativiste développée par le rapporteur [Ann. Sci. Ecole Norm. Sup. (3) **58** (1941), 285–304; MR **7**, 140] dans le cadre de la relativité générale. Soit V_5 la variété fondamentale admettant la métrique $\gamma_{\alpha\beta}$ et le groupe d'isométries à 1 paramètre de générateur infinitesimal ξ. Si l'on adopte dans V_5 le tenseur d'impulsion-énergie $\Theta_{\alpha\beta} = rv_\alpha v_\beta - p\gamma_{\alpha\beta}$ (v vecteur unitaire, p pression propre), les lignes de courant satisfont à un principe d'extremum; de l'invariant intégral relatif $\omega = I_\alpha dx^\alpha$ correspondant, on déduit le tenseur de tourbillon $\Omega_{\alpha\beta}$ rotationnel de I_α. Pour que $\Omega_{\alpha\beta}$ soit invariant par le groupe d'isométries de V_5, il faut et il suffit que $G = I_\alpha\xi^\alpha$ soit constant sur la variété. Pour les mouvements irrotationnels ($\Omega_{\alpha\beta} = 0$), l'A. indique un théorème de Lagrange et pour les mouvements rotationnels (rang de $\Omega_{\alpha\beta} = 2$) une généralisation des théorèmes de Helmholtz. Les résultats sont traduits dans la variété riemannienne quotient V_4 munie de la métrique introduite par F. Hennequin [C. R. Acad. Sci. Paris **239** (1954), 1464–1466; MR **16**, 872]. Une étude concernant les mouvements permanents termine le papier. *A. Lichnerowicz* (Paris).

Fierz, M. Connection between pair density and pressure for a Bose gas consisting of rigid spherical atoms. Phys. Rev. (2) **106** (1967), 412–413.

It is shown that, for a Bose gas of rigid spherical atoms of radius a, the pressure p is given by the pair-density $g(r)$ and the energy per atom u by the formula $p/\rho T = \frac{2}{3}u/T + \frac{2}{3}\pi a^3 \cdot \frac{1}{h}mT \cdot g''(a)$. *Author's summary.*

Widom, B. Statistical mechanics of liquid-vapor equilibrium. J. Chem. Phys. **26** (1957), 887–893.

The configuration integral of a pure fluid is approximated by considering configurations in which the N particles are distributed in two sub-volumes of the volume V, the potential energy per particle in each sub-volume being taken to be a function, $u(v_i)$, of the volume per particle in that region, and interaction between sub-volumes ignored. When all possible distributions of particles between sub-volumes are considered, and when $u(v_i)$ has the form suggested by physical considerations, then the pressure as a function of $v = V/N$ is found to be equal to $\pi(v) = kT/v - u'(v)$ whenever $\pi(v)$ is stable, but to be equal to the constant vapor pressure which satisfies the equal areas criterion whenever $\pi(v)$ is metastable or unstable. The functions $\pi(v)$ themselves have the form of a set of isotherms of van der Waals type. (Author's summary.) *G. Newell* (Providence, R.I.).

Schmidt, Helmut. Disordered one-dimensional crystals. Phys. Rev. (2) **105** (1957), 425–441.

The author gives a new formulation of the eigenvalue problem which yields fairly generally the density of eigenstates for one-dimensional disordered systems. As a first application of the method the frequency spectrum of a disordered linear chain of elastically coupled masses is obtained. Next the method is applied to the problem of calculating the density of electronic energy levels in a one-dimensional disordered crystal characterized by a series of randomly distributed δ-function potentials. Explicit results are obtained in the case that a uniform crystal contains also a small amount of impurities.
H. A. Hauptmann (Washington, D.C.).

Finch, G. I.; and Sinha, K. P. On reaction in the solid state. Proc. Roy. Soc. London. Ser. A. **239** (1957), 145–153.

Denote by T_R the minimum temperature at which powders of a given material begin to sinter, i.e. at which a solid may be expected to enter into a solid-state reaction. The authors employ the phonon-lattice interaction concept and the uncertainty relation to derive an expression for T_R in terms of the characteristic Debye temperature of the solid. For nine of ten listed elements the value of T_R thus calculated satisfies $0.37T_f \leqq T_R \leqq 0.53T_f$, where T_f is the melting point of the solid on the absolute scale, in good agreement with the values experimentally determined by Hüttig [Kolloid-Z. **99** (1942), 262–277].

By means of the quantum rate theory the authors also derive the rate of reaction between two different solids. They conclude that crystallographic phase transformations and the formation of transitional superstructures constitute the phase-boundary processes, that the crystalline framework of the lattices is maintained throughout the course of such a reaction, and that the dynamics of the diffusion process (i.e. the gradients of chemical and electrical potential) determine the reaction rate.
H. A. Hauptmann (Washington, D.C.).

Hori, Jun-ichi; and Asahi, Takashi. On the vibration of disordered linear lattic. Progr. Theoret. Phys. **17** (1957), 523–542.

The authors study the vibrational modes of linear atomic lattices containing atoms of various types. The greater part of the paper is devoted to the study of nearest neighbor interactions, with one section devoted to next-nearest interaction.

The authors consider a system of linear differential equations of the form

$$m\ddot{x}_i=-\lambda[(x_i-x_{i-1})+(x_i-x_{i+1})] \quad (i=1, 2, \cdots, N),$$

with a cyclic boundary condition, where x_i is the displacement of the ith atom, and study the characteristic frequencies of this system under various assumptions concerning the masses. They treat first a monatomic system, where the masses are equal, then a diatomic system, where the odd and even masses have values m_1 and m_2 respectively, then the case where there is one impure atom and then several impure atoms. In addition, they consider a particular type of random lattice [cf. Dyson, Phys. Rev. (2) **92** (1953), 1331–1338; MR **15**, 492; Bellman, ibid. **101** (1956), 19; MR **17**, 930] the case of a hole in a regular lattice, the case on one impure atom in a diatomic lattice, and finally a case of next-nearest interaction. *R. Bellman* (Santa Monica, Calif.).

Glauberman, A. E. Theory of classical systems of interacting particles obeying a noncentral interaction law. I. Soviet Physics. JETP **4** (1957), 169–173.

The theory of systems of interacting particles with noncentral interaction law is considered on the basis of Bogoliubov's method. Successive approximations are obtained for the distribution functions in the two simplest cases: for a gas consisting of axially symmetric diatomic neutral molecules and for a dipole crystal.

From the author's summary.

Temperley, H. N. V. The influence of boundary conditions on the Onsager-Ising partition function for the plane square lattice. Proc. Phys. Soc. Sect. B. **70** (1957), 192–197.

The Kac and Ward treatment of the Ising problem is modified by destroying the cyclic nature of the matrix used in the evaluation of the partition function. The author then shows that various modifications of the boundary conditions do not significantly change the thermodynamic properties at or near the critical temperature. *G. Newell* (Providence, R.I.).

See also: Blanc-Lapierre, p. 949; Landau, p. 975; Das, Roy and Roy, p. 976.

Elasticity, Visco-elasticity, Plasticity

★ **Никифоров, С. Н. [Nikiforov, S. N.] Теория упругости и пластичности. Theory of elasticity and plasticity.]** Gosudarstv. Izdat. Lit. Stroit. i Arhitek., Moscow, 1955. 284 pp. 7.70 rubles.

Thomas, T. Y. Deformation energy and the stress-strain relations for isotropic materials. J. Math. Phys. **35** (1957), 335–350.

The author considers constitutive equations of the form $D\sigma_{\alpha\beta}/Dt=f_{\alpha\beta}(\rho, \sigma_{\mu\nu}, e_{\mu\nu})$ which are invariant under arbitrary time dependent orthogonal coordinate transformations. Here $\sigma_{\alpha\beta}$ is the stress tensor, $\varepsilon_{\alpha\beta}$ the rate of strain and ρ the density. He poses and presents partial solutions of the problem of determining under what conditions there will exist a scalar invariant E of ρ, $\sigma_{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$ such that $\rho dE/dt=\sigma_{\alpha\beta}\varepsilon_{\alpha\beta}$ whenever the constitutive equations are satisfied. *J. L. Ericksen.*

Pogány, B. Les problèmes de l'angle de frottement et de la rhéologie dans la théorie de la poussée des terres. Acta Tech. Acad. Sci. Hungar. **16** (1957), 3–12. (German, English and Russian summaries)

An attempt is made to relate the theory of granular media with that of the linear visco-elastic relaxing (Maxwell) continuum and to express Conlomb's angle of internal friction in terms of a relaxation time. Since the deformation processes in the two media are fundamentally different, any manipulation of phenomenological equations by which their respective parameters can be mutually converted seems physically meaningless.

A. M. Freudenthal (New York, N.Y.).

Szelągowski, Franciszek. A semi-infinite plate having a rigid circular inclusion and subjected to tension. Arch. Mech. Stos. **8** (1956), 695–704.

The author discusses the determination of stresses at a generic point of a semi-infinite plate with a rigid circular inclusion subjected to tensile forces parallel to the edge of the plate. The solution is obtained by conformally mapping the semi-infinite region with a circular hole into a circular ring. The author's calculations show that, for the single case considered, the maximum stress on the boundary of the inclusion is approximately twice the stress at the same point of a similar plate without the inclusion. *W. E. Boyce* (Troy, N.Y.).

Nowiński, Jerzy. The torsion of a bar with cross-section in the form of an annular sector, one cross-section remaining plane. Arch. Mech. Stos. **9** (1957), 73–87. (Polish and Russian summaries)

★ **Salles, Francisque. Contribution à la détermination mathématique approchée des contraintes dans les plaques.** Publ. Sci. Tech. Ministère de l'Air, Paris, no. 321, 1956. vii+161 pp. 2000 francs.

La présente étude est la suite du Bull. Services Tech., Ministère de l'Air, Paris, no. **115** (1951). L'auteur reprend par d'autres méthodes l'étude des contraintes dans une plaque plane, chargée dans son plan, en lui substituant une autre structure où les efforts normaux s'ecoulent par colonnes fictives, le cisaillement par plaques partielles. Une étude sur les plaques longues est donnée, suivi d'un calcul numérique; en utilisant un treillis de colonnes fictives, on est ramené à résoudre des systèmes hautement hyperstatiques que l'on essaye d'orthogonaliser. Des applications sont indiqués: flexion de la poutre creuse, étude des caissons, expressions des déplacements en fonction des contraintes. *R. Gran Olsson.*

Guest, J. The buckling of a clamped parallelogram plate under combined bending and compression. Austral. J. Appl. Sci. **8** (1957), 27–34.

The problem considered is that of the buckling of a clamped parallelogram plate under combined bending and compression. Detailed study of four different types of stress-distribution was made. — Case 1) covers the problem of pure bending of clamped rhombic plates, that is, the two opposite edges are unloaded whilst the other

two are each subjected to linearly varying tension and compression with the midpoint as the zero stress point. Case 2) treats the rhombic plate in pure compression in one direction, the compression being greatest at the lowest boundary and linearly diminishing, reaching zero stress at the top corner. Case 3) deals with an asymmetric stress-distribution in one direction. Here the plate is subjected to linearly varying compression along the lower third of the loaded boundaries and then undergoes linearly varying tension for the remainder of the way up. Galerkin's variation of the method of Rayleigh-Ritz has been used to find the various critical buckling stresses.
R. Gran Olsson (Trondheim).

Prager, W. On the analogy between plates and disks. Proc. Roy. Soc. London. Ser. A. **239** (1957), 394–398.

The analogy between elastic plates and disks of uniform thickness is extended to inelastic plates and disks of varying thickness. The analogy is illustrated by the relation between a disk of varying thickness of Voigt isotropic visco-elastic material and a plate of isotropic visco-elastic material of the Maxwell type. The analogy is of value in experimental work. *A. E. Green.*

Klein, Bertram. Shear buckling of simply supported rectangular plates tapered in thickness. J. Franklin Inst. **263** (1957), 537–541.

The author uses the method of collocation to determine the buckling load in shear of a rectangular plate linearly tapered in thickness. Results are presented in the form of a plot of the buckling coefficient k versus the aspect ratio b/a for various values of the taper parameter α, and of k versus α for several values of b/a. *W. E. Boyce.*

Norbury, J. F. Thermal stresses in disks of constant thickness. Aircraft Engrg. **29** (1957), 132–137.

The author considers elastic stresses due to a radially symmetric temperature distribution in a circular disk of constant thickness. The temperature is assumed to obey a power law with the exponent allowed to vary. Disks with central holes and disks mounted on shafts are discussed, together with continuous disks. The author concludes that the thermal stresses are strongly affected by the temperature profile in the interior of the disk, as well as by the temperature difference between inner and outer radii.
W. E. Boyce (Troy, N.Y.).

Shibaoka, Yoshio. On the buckling of an elliptic plate with clamped edge. II. J. Phys. Soc. Japan **12** (1957), 529–532.

The buckling of a uniform thin elliptic plate with clamped edge is dealt with in case when its periphery is subject to a uniform normal pressure in the plane of the plate, and the relationship between the critical buckling load and the eccentricity of the elliptic plate is shown graphically. *Author's summary.*

Nowacki, Witold. Some stability problems of cylindrical shells. Arch. Mech. Stos. **8** (1956), 705–724. (Polish. Russian and English summaries)

The starting point are V. Z. Vlasov's [General theory of shells ···, Moscow-Leningrad, 1949; MR **11**, 627] two equations for deflection of a cylindrical shell. These are reduced to one equation, which by means of Green's function is represented in form of a Fredholm's integral equation of the second kind with an unsymmetrical kernel. Both the deflection and the Green function are expanded in trigonometric series with the coefficients unknown for the time being. The integral equation is thus reduced to an infinite system of linear algebraic equations. The vanishing determinant of this system furnishes the condition for the instability of the shell. This procedure is applied to some particular cases of a cylindrical shell such as with a) freely supported edges, b) clamped edges, and c) a plate with a longitudinal reinforcing rib. No numerical examples are given. *M. Z. Krzywoblocki* (Urbana, Ill.).

Vyčichlo, F. Beitrag zu einem geometrischen Problem der Schalentheorie. Schr. Forschungsinst. Math. **1** (1957), 286–288.

Remarks on the differential geometrical aspects of the classical, linearised small-deformation, theory of material layers the thickness of which is small compared to representative curvature radii of their bounding surfaces.
E. Reissner (Cambridge, Mass.).

Sharma, Brahma Dev. Stresses due to a nucleus of thermo-elastic strain (i) in an infinite elastic solid with spherical cavity and (ii) in a solid elastic sphere. Z. Angew. Math. Phys. **8** (1957), 142–150.

Solutions, exact within the classical theory of elasticity, are determined for the stresses and displacements induced by a nucleus of thermoelastic strain (center of dilatation) at an arbitrary point of a region bounded, (a) internally and (b) externally, by a sphere. The spherical boundary, in both cases, is assumed to be free from loading. The author starts with the well known results corresponding to a center of dilatation at a point of a medium occupying the entire space, and subsequently removes the ensuing surface tractions with the aid of axisymmetric solutions of the field equations introduced previously by the reviewer, R. A. Eubanks, and M. A. Sadowsky, [Proc. 1st U.S. Nat. Congress Appl. Mech. Chicago, 1951, Amer. Soc. Mech. Engrs., 1952, pp. 209–215; MR **14**, 926]. The complete solutions thus obtained involve infinite series of Legendre functions. Numerical values are given for the boundary stresses and displacements.

The two special problems under consideration are prerequisites to the treatment of the general thermoelastic problem (arbitrary temperature field) for the two regions at hand by means of Goodier's method of integration. In view of the unwieldiness of the present results, their significance in this broader context is, unfortunately, quite limited. *E. Sternberg* (Providence, R.I.).

Choudhury, Pritindu. Stress distribution in a thin aeolotropic strip due to a nucleus of strain. Z. Angew. Math. Mech. **36** (1956), 413–416. (English, French and Russian summaries)

Formal expressions are obtained for the stress distribution due to a nucleus in the form of a centre of dilatation situated midway between the straight edges of a long thin aeolotropic strip. *A. E. Green.*

★ **Neal, B. G. The plastic methods of structural analysis.** John Wiley & Sons, Inc., New York, 1956. xi+353 pp. $7.50.

The first four chapters of this book deal respectively with basic hypotheses, simple cases of plastic collapse, basic theorems, and general methods for plastic design of framed structures. In this presentation, the fundamental concepts of equilibrium and compatibility are stressed, and the connection between a collapse mechanism of a

structure and a corresponding equilibrium equation is established by use of the principle of virtual work. Proofs of the theorems stated in Chapter 3 are given in an appendix; the theorems are applied in Chapter 4 to the derivation of several alternative (and in some cases complementary) methods of analysis of real frames. These four chapters could well form the basis for a graduate or final year undergraduate course in the plastic theory of structures; indeed, the student who masters these chapters will have a clear understanding not only of the plastic behaviour of structures, but also of the general principles of engineering structures, whether elastic or plastic.

The last four chapters apply plastic methods to four separate topics. Chapter 5 gives an account of the estimation of deflections at collapse of a structure. Chapter 6 discusses factors which may effect the value of the full plastic moment of a member, including those of shear force and axial load. (The author does not discuss the promotion of instability by axial load, but confines himself to structures for which it is assumed that plastic collapse occurs before the onset of instability. Thus the book cannot be used by itself for practical design purposes, but references are given to work on column buckling and lateral stability.) Chapter 7 deals with minimum weight design, and Chapter 8 with shakedown under variable repeated loading.

Numerical examples to be worked by the reader are given at the end of each chapter, and the references are comprehensive. *J. Heyman* (Providence, R.I.).

Keller, W. Neue kritische Drehzahlen von einfach besetzten Wellen. Ing.-Arch. **25** (1957), 71–89.

A heavy wheel is eccentrically mounted on a weightless shaft and rotated at frequency $\omega/2\pi$. Compressional and torsional forces of the form $A+B\cos\nu\omega t$ (ν an integer) are applied. The equations of motion are completely linearized and reduced to a non-homogeneous Mathieu equation with periodic forcing term. The resonances (i.e., values of ω for which the lateral displacement of the wheel is unbounded) are investigated, largely by means of formal power series developments. *H. F. Weinberger.*

Suppiger, Edward W.; and Taleb, Nazih J. Free lateral vibration of beams of variable cross section. Z. Angew. Math. Phys. **7** (1956), 501–520.

The standard procedure for obtaining normal modes of vibration is carried out for a beam with exponentially varying mass density and section moment and constant radius of gyration. Examples are given for various boundary conditions and typical shapes of tubes which satisfy these conditions are illustrated. *E. H. Lee.*

Fogagnolo Massaglia, Bruna. Sulle vibrazioni trasversali di uno strato elastico libero sulle basi ed appoggiato sui bordi. Univ. e Politec. Torino. Rend. Sem. Mat. **15** (1955–56), 163–174.

L'argomento, quale appare enunciato nel titolo e poi in modo più particolareggiato nel primo paragrafo introduttivo, tende alla determinazione delle possibili vibrazioni di una piastra rettangolare di spessore non trascurabile, ma tuttavia sensibilmente minore che le altre due dimensioni (Nella terminologia di A. E. H. Love "moderately thick plates"). Rapporti precisi tuttavia non si assegnano. Si svolge di fatto la ricerca nella pura forma di risoluzione di un problema di analisi differenziale sotto convenienti ipotesi essenzialmente semplici. *B. Levi.*

Chu, Hu-Nan; and Herrmann, George. Influence of large amplitudes on free flexural vibrations of rectangular elastic plates. J. Appl. Mech. **23** (1956), 532–540.

Nonlinear equations of plate vibration are used which include the influence of membrane stresses in the form of the von Karman large deflection equilibrium theory. A perturbation analysis is developed for vibration of a rectangular plate with simply supported edges in terms of the small parameter (thickness/length). By assuming a space variation of deflections compatible with the equations and boundary conditions, a nonlinear vibration equation for the central deflection is obtained which can be solved in terms of elliptic integrals. Examples are given for the variation of period and maximum stress with amplitude of oscillation. Other methods of approximation are considered and the equivalent analysis for a beam is given for comparison. *E. H. Lee* (Providence, R.I.).

Hodge, P. G., Jr.; and Perrone, Nicholas. Yield loads of slabs with reinforced cutouts. J. Appl. Mech. **24** (1957), 85–92.

The techniques of limit analysis are used to determine upper and lower bounds to the load carrying capacities of square slabs with central cut-outs. The slabs are made of ductile metal, and are subjected to 'uni-axial tensile loading' parallel to one pair of opposite edges. The practical conditions of this type of loading are not always precisely known, but are intermediate to the extremes of uniform distributions of normal stress or normal velocity along the edges concerned. In the present paper, both these assumptions are considered. Results are obtained for various cases of square and circular cut-outs with or without reinforcement at the cut-out edge. Plane stress conditions are assumed in the slab, and the reinforcement is treated according to simple curved beam theory. The Tresca yield condition and associated flow rule are adopted for obvious reasons of simplicity. The effect of transverse shear force on the yielding of the reinforcement is neglected. Importantly, a detailed comparison is made between theoretical values of load carrying capacities and experimental values of failure loads for steel slabs reported by D. Vasarhelyi and R. A. Hechtman [Welding J. Supplement **30** (1951), 182s–192s]. It is found that the assumption of uniform normal velocity along the loaded edges leads to quite good agreement between experiment and theory. The paper concludes with the suggestion that appropriately factored upper bounds (the lower bounds being laborious to compute) should be used for design purposes. *H. G. Hopkins* (Sevenoaks).

Niepostyn, Dionizy. The limit analysis of an orthotropic circular cylinder. Arch. Mech. Stos. **8** (1956), 565–580.

The paper is concerned with the plastic behavior of cylindrical shells made of a rigid-plastic material which is orthotropic with respect to axial and circumferential directions. The material is assumed to obey a modified version of the Tresca yield condition and the associated flow rule. Under these assumptions the yield condition for the stress resultants is found using a previously developed method for isotropic shells [Onat, Quart. Appl. Math. **13** (1955), 63–72; MR **16**, 977].

The mathematical problem of finding the critical load intensity is considered for various examples. It is to be pointed out that Fig. 9 of the paper describing the yield curve for the zero axial force is not correct.

E. T. Onat (Providence, R.I.).

Bölcskei, E. The limit load carrying capacity of compression bars made of perfectly plastic materials. Acta Tech. Acad. Sci. Hungar. **17** (1957), 3–23. (German, French, and Russian summaries)

The paper is concerned with the load carrying capacity of a beam-column of rectangular cross section that is subjected to an eccentric compressive load. The longitudinal fibers are assumed to behave in an elastic, perfectly plastic manner and fail when the strain reaches a critical absolute value. The beam-column is considered to have reached its load carrying capacity when the critical strain is attained in a surface fiber of some cross section.
W. Prager (Providence, R.I.).

Sawczuk, Antoni. Some problems of load carrying capacities of orthotropic and non-homogeneous plates. Arch. Mech. Stos. **8** (1956), 549–563.

The load carrying capacities of isotropic circular plates under rotationally symmetric loading were first analyzed by Hopkins and Prager [J. Mech. Phys. Solids **2** (1953), 1–13; MR **15**, 270]. In the present paper, this analysis is extended to orthotropic circular plates under rotationally symmetric loading, the principal directions of orthotropy being radial and circumferential. Minimum weight design of plates of this kind is also discussed. *W. Prager.*

Kačanov, L. M. On the problem of stability of elastoplastic equilibrium. Vestnik Leningrad. Univ. **11** (1956), no. 19, 114–132. (Russian)

In the introduction Kačanov discusses the present status of criteria of stability arriving at the conclusion that static criteria of stability are appropriate for conservative systems, whereas for non-conservative ones the dynamic criteria should be used. A review of existing hypotheses including Engesser-Kármán's and others, shows that in the elastoplastic range only Shanley's criterion seems to be appropriate. Assuming this as the basic hypothesis, Kačanov derives in the first part equations of stability of elasto-plastic equilibrium for bars and rods. Starting from the formulas for deformations in the plastic range, Kirchhof's differential equations of equilibrium jointly with Clebsch's relations are applied. After some manipulations there result two final ordinary differential equations of the fourth order for the displacements, whose particular case are Greenhell's equations. In the second part this technique is applied to thin-walled bars. For illustrative purposes a double T-section is considered as a special case. In the third part the analogous equations for plates and cylinders are derived. No numerical examples are calculated. The list of references contains extensive American literature. *M. Z. Krzywoblocki.*

Lin, T. H. Analysis of elastic and plastic strains of a face-centred cubic crystal. J. Mech. Phys. Solids **5** (1957), 143–149.

The effect is analyzed of the elastic strain on the plastic deformation by multiple slip along five independent slip systems of a single face-centered cubic crystal under uni-axial strain. The investigation represents an extension of G. I. Taylor's classical work in which the crystal was assumed to remain rigid until slip started simultaneously on the critical combination of five active slip systems; it permits the study of possible slip-sequences on the five active systems and shows that the elastic strain plays an important role in slip initiation.
A. M. Freudenthal (New York, N.Y.).

See also: Campanato, p. 905; Riparbelli, p. 966; Sivuhin, p. 967; Das, Roy and Roy, p. 976.

Fluid Mechanics, Acoustics

Milne-Thomson, L. M. Some hydrodynamical methods. Bull. Amer. Math. Soc. **63** (1957), 167–186.

This is a review article, so that its detailed contents need not be noticed here. One may comment, however, on the following introductory remarks, designed obviously enough to attract discussion: "Mathematics is about the logical consequences of assumed propositions, nowadays called axioms. Thus all mathematics is one. The fancied distinction between 'pure' and 'applied' is a modern and false dichotomy unknown to Euler and Cauchy."

The reviewer would suggest that Euler and Cauchy were universalists because they did not restrict themselves to the construction of logical edifices on axiomatic foundations, but rather had wider interests, including the development, assisted by mathematical method, of the delicate art of making fruitful physical hypotheses.
M. J. Lighthill (Manchester).

Fel'zenbaum, A. I. Investigation of vortex motions of a fluid by the methods of analytic functions with a perfect set of singular points. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **11** (1956), no. 1, 17–22. (Russian)

The present application of the theory of complex analytic functions to the plane flow of an incompressible fluid admits only the existence of isolated vortexes (singular points). But if in the vector field of flow there exist whole vortex regions then the classical approach cannot be used. To solve this problem Fel'zenbaum uses the tool of complex analytic functions with a perfect set of singular points. Assume a plane motion of an incompressible fluid in a domain Σ and a perfect set S of singular points in Σ, corresponding to vortexes. The vortex intensity γ is a continuous function of coordinates in S and its value in S is nowhere equal to zero. The complex potential $w(z)$ is equal to the surface integral over S of the product of $[\gamma(t)\ln(z-t)]$ divided by $2\pi i$, where t refers to the values in S only. It expresses the magnitude of w everywhere in Σ outside a small region around the point z. Using the tool of the theory of analytic functions, Fel'zenbaum shows that: (a) the complex velocity $f(z)=w'(z)$ is an analytic function in the whole plane with an everywhere perfect set of singular points; (b) all the points of S are singular points of $f(z)$; (c) $f(z)$ is bounded in the whole plane; (d) $w(z)$ is an analytic function. Next Fel'zenbaum derives both the generalized Blasius formulae for the force and moment (also for the arm of the force) acting on a closed contour (body) in a uniform flow. In agreement with the intuitive guess the circulation is shown to be equal to the surface integral over S of $\gamma(t)$.
M. Z. Krzywoblocki (Urbana, Ill.).

Lighthill, M. J. Corrigenda to "Drift". J. Fluid Mech. **2** (1957), 311–312.

The author points out some errors in same J. **1** (1956), 31–53 [MR **18**, 437] and corrects them. The most serious of these concerns the calculation of the velocity due to trailing vorticity. In addition, there was a calculation of the so-called tertiary flow, which the author now criticizes as incomplete. He promises a new paper in the same journal. *W. R. Sears* (Ithaca, N.Y.).

Case, K. M.; and Parkinson, W. C. Damping of surface waves in an incompressible liquid. J. Fluid Mech. 2 (1957), 172–184.

The damping of surface waves of small amplitude in a slightly viscous liquid contained in a cylinder is calculated approximately. It is found that most of the damping comes not from the interior of the liquid but from assumed boundary layers adjacent to solid fixed boundaries, and values for the damping decrement are calculated. The authors have made experiments to test these calculations, thereby adding greatly to the interest of their paper. It was found that the observed logarithmic decrement for motion in an unpolished brass tube was too large by a factor of between 2 and 3, but that good agreement with theory was obtained when the tube was polished to a mirror finish. It is stated that in the unpolished tube the depth of the roughnesses was small compared to the boundary layer thickness, and the effect of polishing is thus very surprising. The reasons for this are not yet understood. It was also found that the observed frequency was a few percent higher in most cases than the calculated frequency; the authors suggest that the wetting of the wall near the free surface may be partly responsible but are as yet unable to give a quantitative explanation.

{The reviewer believes that this problem deserves further experimental study. There is evidence from elsewhere that impurities can greatly increase the dissipation by forming thin surface layers, and strict precautions will be needed to deal with these.} *F. Ursell.*

Welander, Pierre. Wind action on a shallow sea: Some generalizations of Ekman's theory. Tellus 9 (1957), 45–52.

The author presents a theory for the sea level changes in shallow water produced by a steady wind. It is shown that the wind stress divergence can be as important as the wind stress curl. The transient case is also examined and a single integro-differential equation is derived for the sea level elevation. *H. Greenspan.*

Cicala, Placido. Soluzioni discontinue nei problemi di volo ottimo. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 90 (1955–56), 533–551.

Kästner, Siegfried. Die Differentialgleichungen der Strömung einer inkompressiblen Newtonschen Flüssigkeit in einem Wendelkanal. Z. Angew. Math. Mech. 37 (1957), 148–149.

Hill, Jacques A. F. Integral methods for laminar forced convection calculations - an evaluation of two approaches. Papers on heat transfer and unsteady aerodynamics presented at the IX International Congress of Applied Mechanics, Brussels, Belgium. Naval Supersonic Laboratory, Massachusetts Institute of Technology, Tech. Rep. 179 (1956), 16 pp.

This is a valuable review of methods of laminar heat transfer calculation at small Mach number and small relative temperature difference. Integral methods and Reynolds-analogy methods are reviewed, and improvements suggested. It is shown how an improved form of a formula of the reviewer can be obtained by a suitable integral method. Different approximate methods are evaluated by comparison with an exact computation by Frössling [Lunds Univ. Årsskrift (N.F.) 36 (1940), no. 4; MR 2, 331]. Integral methods come well out of the comparison, but methods based on the so-called Reynolds analogy, which is a false analogy in the presence of pressure gradients, are shown to be seriously inaccurate. *M. J. Lighthill* (Manchester).

Krzywoblocki, M. Z. On the application of successive approximations to motion started impulsively from rest in compressible media. Acad. Serbe Sci. Publ. Inst. Math. 9 (1956), 41–58.

Some of the equations governing the title problem are written down and ineffectually discussed. *M. J. Lighthill* (Manchester).

Riparbelli, Carlo. Una caratteristica dei moti stabili di fluidi reali. Aerotecnica 37 (1957), 13–23.

Author applies extremum energy principles to viscous flow. The material is unoriginal and of undergraduate level. *D. R. Bland* (Manchester).

Mattioli, Ennio. Una formula universale per lo spettro nella turbolenza di parete. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 90 (1955–56), 298–310.

Legendre, Robert. Singularités critiques de l'hodographe d'un écoulement plan. C. R. Acad. Sci. Paris 244 (1957), 851–852.

The author remarks that the regular solutions of the compressible flow hodograph equations are not sufficient to define all of the possible flows in the physical plane. He points out that solutions with critical points in terms of the hodograph variables may represent regular flows in the physical plane, the critical points being the image of stationary points for the velocity. *Hirsh Cohen.*

Cole, Julian D.; and Messiter, Arthur F. Expansion procedures and similarity laws for transonic flow. I. Slender bodies at zero incidence. Z. Angew. Math. Phys. 8 (1957), 1–25.

Beginning with the equations of steady inviscid flow past a solid body and the shock-wave relations, the authors propose to introduce expansions of the physical variables in terms of a (small) thickness parameter. A coordinate distortion is also assumed, in order that the respective velocity components have the correct orders of magnitude for infinitesimal thickness. Requirements for the success of such an expansion procedure are set down; they are, briefly, (i) that the shock-wave relation should not degenerate, and (ii) that the differential equations should apply in the large. Flow past a body of revolution is considered in particular. To the first order, the von Kármán theory is found; viz., the axial and radial velocity perturbations are proportional to δ^2 and δ^3, respectively where δ is the body's thickness ratio.

To proceed to the second order it is found necessary to consider the nature of the first-order solution for slender bodies of revolution near the axis. This deduction is carried through and it is concluded that the second-order terms are proportional to $\delta^4 \ln \delta$ and $\delta^5 \ln \delta$, respectively. The third-order terms, proportional to δ^4 and δ^5, are also found. The flow is found to be irrotational to third order. The equation for the first-order potential is, of course, the familiar nonlinear one of von Kármán [J. Math. Phys. 26 (1947), 182–190; MR 9, 217]. The equations for the higher-order potentials are linear with variable coefficients.

Surface pressures and drag are next computed, especially for bodies of nearly circular cross-section. The result is a sort of "area rule" in which the drag of the non-circular

body is compared with that of a body of revolution with the same cross-sectional area distribution. The difference in drag of the two bodies is shown to be $o(\tau^2\delta^2)$, where τ is the order of the perturbations from circular cross-section shape. *W. R. Sears* (Ithaca, N.Y.).

Bagchi, R. N. On an extension of Kar formula of minimum bowing pressure. Indian J. Theoret. Phys. **4** (1956), 15–19.

Kar's formula for minimum bowing pressure to produce a musical note [same J. **2** (1954), 46] is extended by calculating the frictional force more rigorously. The formula thus obtained satisfactorily explains both the straight portion (at high velocities) and the upward bend (at low velocities) of the $P_{\min}-V$ curve. *Author's summary.*

Miles, John W. On the reflection of sound at an interface of relative motion. J. Acoust. Soc. Amer. **29** (1957), 226–228.

Ramakrishna, B. S. Modes of vibration of the Indian drum dugga or left-hand thabala. J. Acoust. Soc. Amer. **29** (1957), 234–238.

Alblas, J. B. On the diffraction of sound waves in a viscous medium. Appl. Sci. Res. A. **6** (1957), 237–262.

The author obtains a complete mathematical solution to the problem of diffraction by a half-plane barrier of harmonic sound waves of infinitesimal amplitude in a gas, when shear viscosity is taken into account but not heat conduction or bulk viscosity (although these last two effects could actually be taken into account as well without significantly complicating the mathematics). The boundary condition at the half-plane barrier being that of zero velocity, there is a boundary layer within which departures from the classical Sommerfeld solution (which assumed merely zero normal velocity) are large. This boundary layer, like others arising in motions with frequency ω, has a thickness δ of order $\sqrt{(\nu/\omega)}$, where ν is the kinematic viscosity. It leads to a correction term in the far-field behaviour which is of order δ/λ compared with the main scattered-field term (where λ is the wavelength). The mathematical solutions are effected by means of the Wiener-Hopf technique. *M. J. Lighthill.*

Jones, D. S. Approximate methods in high-frequency scattering. Proc. Roy. Soc. London. Ser. A. **239** (1957), 338–348.

The high-frequency scattering of a plane wave by two-dimensional obstacles without sharp edges is treated. The wave equation is $(\nabla^2+k^2)\phi=0$, and the boundary condition on the obstacle is $\phi=0$ or $\partial\phi/\partial n=0$. The problem is to find an asymptotic expression for the total scattering coefficient as $ka\to\infty$, where a is a typical dimension of the obstacle. For a circle of radius a the total scattering coefficient σ is known to be of the form

$$\text{(A)}\qquad \sigma\sim 2+\frac{p}{(ka)^{2/3}}+\cdots,$$

where p is a number depending on the boundary condition; existing proofs of this are complicated. The purpose of the present paper is to give plausible but simple arguments in support of this result and to extend it to more general obstacles.

The coefficient σ can be expressed in many ways as a line integral involving ϕ, but if ϕ is known to a given degree of accuracy, the different integrals for σ do not all define σ with equal efficiency. In the present paper the author first derives for σ an efficient line integral applicable to a class of obstacles with symmetry. This he applies to the special case of the circle. He substitutes for ϕ (and its derivatives) the values given by geometrical acoustics, except near the shadow boundary where geometrical acoustics is inapplicable and where he uses two different approximations for ϕ. The first one gives an expression for σ of the correct form (A) but with coefficients p a few percent in error. The second, somewhat more complicated, gives the correct values of p. The author suggests that analogous formulas hold for general non-circular convex obstacles. *F. Ursell.*

Sivuhin, D. V. Diffraction of a plane sound wave on a spherical cavity. Akust. Ž. **1** (1955), 78–88. (Russian)

The author deals with the problem of scattering of a plane monochromatic acoustic wave by a spherical hole in an infinite uniform elastic medium. A scalar and a vector potential are introduced, each satisfying a wave equation. The scalar potential contributes to asymptotically longitudinal waves, while the vector potential contributes to asymptotically transverse waves. The problem is solved by expansion into spherical harmonics. The coefficients of the first two terms are calculated as a function of size of the cavity. The resonance shown by the curves is discussed. *J. Shmoys* (Brooklyn, N.Y.).

See also: Lewy, p. 883; Ladyženskaya, p. 901; Itô, p. 941; Kampé de Feriet, p. 949; Chandrasekhar, p. 969; De, p. 970; Longe, p. 977.

Optics, Electromagnetic Theory, Circuits

Cowley, J. M.; and Moodie, A. F. Fourier images. I. The point source. Proc. Phys. Soc. Sect. B. **70** (1957), 486–496.

The authors discuss the imaging of a periodic object (a plane grating) by an optical or electronic system, using the methods and language of information theory. They treat especially the case in which the grating is illuminated by a point light source, in which case there exist multiple focal planes, giving an interference image of the grating. The author calls these images, "Fourier images." The problem is similar to the one investigated in Abbe's theory of the microscope; the mathematical methods, however, are different and the results are applied to electronic systems. *M. Herzberger* (Rochester, N.Y.).

Cowley, J. M.; and Moodie, A. F. Fourier images. II. The out-of-focus patterns. Proc. Phys. Soc. Sect. B. **70** (1957), 497–504.

In the second paper, the authors investigated the Fourier images of a diffraction grating illuminated by a point light source in other than a focal plane. Experimental investigations show that with an angle of fifteen degrees, the approximation formulae of Fresnel are satisfactory. *M. Herzberger* (Rochester, N.Y.).

Cowley, J. M.; and Moodie, A. F. Fourier images. III. Finite sources. Proc. Phys. Soc. Sect. B. **70** (1957), 505–513.

In the third paper, the authors discuss the interference image of a periodic grating illuminated by a pair of point

sources and by a rectangular source. The results are experimentally tested and applied to the electron microscopy of crystals. *M. Herzberger* (Rochester, N.Y.).

Olivier, Jean. La visibilité des sources lumineuses dans le brouillard. Effet des luminances de la lumière naturelle. Rev. Opt. **36** (1957), 105–131.

A study of the luminance in various directions of a horizontally infinite layer of fog, illuminated by the sun. It is shown that the exact form of the polar scattering function has very little effect on the results except near the top of the fog. The theory is applied to the visibility of the light sources seen horizontally and obliquely, with the result that for a uniform fog there is very little difference between the two except for observations against the sun from near the top of the fog. *W. E. K. Middleton.*

Laudet, Michel. Calcul de l'induction et de ses dérivées sur l'axe d'une lentille électronique magnétique de révolution. C. R. Acad. Sci. Paris **243** (1956), 1855–1857.

In electron microscopes it is important to know the axial magnetic induction $B(z)$ of a magnetic lens in order to deduce its geometrical optical properties. The usual procedure is to employ the method of successive iterration in obtaining the solution of the Laplace equation of the magnetic flux $\varphi(\rho, z)$ when its value is given on the circumference ot a circle $\rho=a$. The induction is found from the relation $B(z)=(2\pi)^{-1}\varphi_{\rho\rho}$ $(\rho=0)$ by means of numerical differentiation for each value of $\varphi(a, z)$. The author considers the boundary value problem $\varphi(a, z)=0$ for $r\neq 0$, $\varphi(a, 0)=1$ for $r=0$, where $r=\sqrt{(\varphi^2+z^2)}$. If the values of the induction at the nodes of a net N_i are denoted by b_i which determine the function $\varphi(\rho, z)$, then the value of the induction B_0 at N_0 is expressed by the successive derivatives of the induction B along the axis. The values of φ_i at the nodes of the net are tabulated for various values of the ratio z/a and the coefficients b_i and its derivatives by numerical differentiation from the values of the flux function. The induction B is expressed in terms of the flux by the following relation:

$$B_0=b_0\varphi_0+b_1(\varphi_1+\varphi_{-1})+b_2(\varphi_2-\varphi_{-2})+\cdots.$$

The relative errors for B and its two derivatives B', B'' calculated on the axis of a coil of radius c from the values of $\varphi(a, z)$ on a cylinder of radius $a=\frac{1}{2}c$ are found to be less than 0.2×10^{-2}, 0.8×10^{-2} and 1.0×10^{-2} respectively. *N. Chako* (New York, N.Y.).

Lenz, Friedrich. Zur Berechnung von rotationssymmetrischen Potentialfeldern in Elektronenlinsen. Ann. Physik (6) **19** (1956), 82–88.

In this paper the author gives a procedure for calculating certain constants which enter in the formula of the potential distribution along the axis of a rotational symmetric three electrode lens. The outside electrodes are maintained at equal potential and are equidistant from the inner electrode. The aperture of the inner electrode is smaller than of the outer electrodes. By introducing a potential function which gives the correct boundary values at large distances from the electrodes an expression for the axial potential distribution of the system has been obtained, from which simple relations are derived for the two arbitrary constants entering in this formula. Furthermore, the author points out the physical and mathematical ambiguity of the formula obtained by Regenstreif [Ann. Radioélec. **6** (1951), 51–83, 114–155] for determining these constants. The author has treated also the case where the inner electrode has an appreciable thickness and also for a system formed by two cylindrical electrodes of different diameters. The results of calculations for the dependence of the field maxima and halfwidth with respect to the ratio of the cylindrical electrode diameters are compared with measured values and are found to be in agreement with the latter.
N. Chako. (New York, N. Y.).

Akhiezer, A. I.; and Polovin, R. V. Theory of wave motion of an electron plasma. Soviet Physics. JETP **3** (1956), 696–705.

The properties of wave motion in an infinitely extended plasma are studied with a model in which the positive charges form a homogeneous background, the density fluctuations arising from the electrons alone. The electron gas is described in a hydrodynamic sense, all electromagnetic variables, as well as the electron momentum, being functions of a single set of space and time variables. The field equations are used in complete form, so that the treatment is not restricted by the limitations imposed in magnetohydrodynamic theory, but is a kind of extension of the usual theory of the optical properties of matter. Solutions are assumed in the form of plane waves and their velocity and polarization properties are studied under various conditions, for both longitudinal and transverse waves. {The reader should be on his guard for mathematical errors. Equations (10) and (12) should should have $\mathbf{i}\cdot\mathbf{H}$ instead of $\mathbf{i}\cdot\mathbf{H}_0$ on the right hand side. This error, which appears in the original article [Z. Eksper. Teoret. Fiz. **30** (1956), 915–928] seems to be fundamental since with this correction Eq. (10) is not completely soluble for $\mathbf{H}$, while in the form given the right hand side of (12) should vanish identically and the equation be integrable for $\mathbf{p}$. Equation (13) is incorrectly transcribed from the original article, but the reviewer has not been able to verify the form given there. The reviewer has not attempted to determine the extent to which these slips may invalidate the conclusions of the paper.}
E. L. Hill (Minneapolis, Minn.).

★ **Гольдштейн, Л. Д.; и Зернов, Н. В. [Gol'dštein, L. D.; i Zernov, N. V.] Электромагнитные поля и волны [Electromagnetic fields and waves.]** Izdat. "Sovetskoe Radio", Moscow, 1956. 639 pp. 16.50 rubles.

Textbook to be used in courses for radio technicians. Contains a systematic presentation of the theory of the electromagnetic fields starting from elementary phenomena, including Maxwell's equations and emphasizing rapidly changing fields.

Gurevich, A. V. On the effect of radio waves on the properties of plasma (ionosphere). Soviet Physics. JETP **3** (1957), 895–904.

Scott, E. J. Wave propagation in a stratified medium. Ann. Polon. Math. **3** (1957), 213–217.

The author considers the problem of the propagation of waves in a stratified medium. More particularly, the Laplace transform method is used to find the voltage in a dissipationless transmission line consisting of n segments with corresponding line constants a_k. The differential equations to be solved are $\partial^2V_k/\partial t^2=a_k^2\partial^2V_k/\partial x^2$ $(x_{k-1}<x<x_k;\ x_0=0;\ t>0;\ k=1, 2, \cdots, n)$ subject to general boundary and initial conditions and the matching

of the functions and their gradients across the ends of adjacent segments. The theory of residues is used to evaluate the inversion integrals.

A particular example of a finite transmission line, initially in equilibrium, consisting of two segments in which the ends $x=0$ and $x=x_2$ are maintained at zero potential and constant potential E_0, is solved explicitly.

It is remarked that the same technique also applies to a spherically stratified medium. *C. G. Maple.*

Wheelon, Albert D. Relation of radio measurements to the spectrum of tropospheric dielectric fluctuations. J. Appl. Phys. **28** (1957), 684–693.

Expressions are derived relating the physically observable parameters of wave propagation to the spectrum of fluctuations of the dielectric constant. *C. H. Papas.*

Levin, M. L. Thermal radiation of good conductors. Soviet Physics. JETP **4** (1957), 225–236.

Radiation from good conductors is examined by the methods of the electrodynamical theory of thermal fluctuations. In the first part of the work, radiation in the wave zone is found, with particular attention devoted to the limiting cases of very short and very long waves. Radiation from bodies with surface anistropy is examined. In the second part of the work, the fluctuating field in the neighborhood of conducting surfaces is considered: near a metallic plane, at the focus of a parabolic mirror, and at the center of a spherical mirror. Fluctuating surface charges are calculated. *From the author's summary.*

Becker, C. H. Radiation magnetization. Z. Physik **148** (1957), 391–401.

Wait, James R. Amplitude and phase of the low-frequency ground wave near a coastline. J. Res. Nat. Bur. Standards **58** (1957), 237–242.

Gutshabash, S. D. The scattering of radiation in the two-layer atmosphere. Vestnik Leningrad. Univ. **12** (1957), no. 1, 158–164, 211. (Russian. English summary)

Woods, Betty D. The diffraction of a dipole field by a half-plane. Quart. J. Mech. Appl. Math. **10** (1957), 90–100.

The problem of the diffraction of an arbitrarily orientated electromagnetic dipole by a perfectly conducting half-plane is solved by extending a method due to Bromwich [Proc. London Math. Soc (2) **14** (1915), 450–463], which was based on the solution of a scalar problem and valid only when the axis of the dipole is parallel to the edge of the screen. This method leads to a field which satisfies Maxwell's equations and the appropriate boundary conditions on the screen: but this solution satisfies the correct edge conditions only when the axis of the dipole is parallel to the edge.

To overcome this difficulty, the author adds to the Bromwich field another field which satisfies Maxwell's equations everywhere except on the screen and satisfies the boundary conditions on the screen: this additional field is chosen so as to make the total field satisfy the correct edge conditions. The details are worked out in the following two cases: (i) an electric dipole with axis normal to the screen; (ii) an electric dipole with axis parallel to the screen and normal to the edge. From these two solutions and the Bromwhich solution with the axis parallel to the edge, the solution for any orientation of the dipole can be found. *E. T. Copson* (St. Andrews).

Steel, W. H. Effects of small aberrations on the images of partially coherent objects. J. Opt. Soc. Amer. **47** (1967), 405–413.

The author studies the diffraction images of periodic gratings and of single lines in the presence of small aberrations. As an example of imaging a coherent and an incoherent object, the theory is extended to the study of two-dimensional objects. *M. Herzberger.*

Wait, James R.; and Mientka, Walter. Slotted-cylinder antenna with a dielectric coating. J. Res. Nat. Bur. Standards **58** (1957), 287–296.

The authors present numerical results for the radiation field of a metallic cylinder uniformly coated with a layer of dielectric and excited by a narrow axial slot.

C. H. Papas (Pasadena, Calif.).

Schelkunoff, S. A. Conversion of Maxwell's equations into generalized telegraphist's equations. Bell System Tech. J. **34** (1955), 995–1043.

The author shows how Maxwell's equations in a waveguide (in the general sense, i.e. not necessarily a cylindrical region and not necessarily with perfectly conducting walls) can be reduced, by using a suitable basis in the cross-section, to a set of generalized telegraphist's equations

$$\frac{\partial V}{\partial z}=-ZI-{}^{V}TV,\ \frac{\partial I}{\partial z}=-YV-{}^{I}TI,$$

where V and I are infinite dimensional vectors and Z, Y, ${}^{V}T$, ${}^{I}T$ are matrices. If the basis is well chosen, then, for the mode of interest, all but one voltage coefficients are small and a perturbation procedure can be used. A number of interesting examples are discussed in detail.

J. Shmoys (Brooklyn, N.Y.).

Kacenelenbaum, B. Z. Irregular wave guides with slowly changing parameters. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 711–714. (Russian)

The paper deals with the propagation of transverse electric waves in a waveguide of variable cross-section. The field in the waveguide is expanded into a set of modes (locally) determined by the local cross-section. If the cross-section is varying slowly then the coefficients of the incident and reflected waves satisfy a set of linear first order ordinary differential equations. A perturbation procedure can then be used to solve this set of equations.

J. Shmoys (Brooklyn, N.Y.).

Skal'skaya, I. P. The electromagnetic field of a dipole radiator placed inside a parabolic reflector. Ž. Tehn. Fiz. **25** (1955), 2371–2380. (Russian)

The paper deals with the radiation of a dipole located at the focus of a perfectly conducting paraboloid of revolution and perpendicular to its axis. An integral representation of the field is obtained and its asymptotic behavior for short wavelengths is investigated. The high frequency limit is shown to agree with the result of a geometric optical calculation. *J. Shmoys.*

Chandrasekhar, S. Hydromagnetic oscillations of a fluid sphere with internal motions. Astrophys. J. **124** (1956), 571–579.

Previous work on the oscillations of a fluid sphere in the presence of a magnetic field has proceeded from the assumption of a static equilibrium field. The present investigation takes as its starting point the theorem

[Chandrasekhar, Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 273–276; MR 18, 357] that for an inviscid, incompressible fluid of infinite electrical conductivity there exist steady solutions of the basic equations, stable against small disturbances, in which the velocity field is everywhere proportional to the magnetic field:

$$\mathbf{v}=\frac{\mathbf{H}}{\sqrt{(4\pi\rho)}}.$$

By dropping the condition of static equilibrium, and using instead the condition that the undisturbed velocity and magnetic fields are stationary and satisfy the above condition, one gains enormous freedom in the choice of the undisturbed field. This paper treats in detail the case when the undisturbed fields are purely toroidal. The author develops a variational expression for the periods of small oscillations about the equilibrium configuration in this case, and uses the Rayleigh-Ritz method to obtain lower bounds for the periods of the two lowest modes of vibration about an equilibrium configuration in which the fields are described by the (0, 1) toroidal mode.

D. Layzer (Cambridge, Mass.).

Cole, G. H. A. Some aspects of magnetohydrodynamics. Advances in Physics 5 (1956), 452–497.

This review article treats the following topics in magnetohydrodynamics: The basic equations and the particular forms they assume under various simplifying assumptions; wave motion in compressible and incompressible media; thermal instability in the presence of a magnetic field; the propagation and structure of magnetohydrodynamic shock waves; and, very briefly, experimental work and astrophysical applications. For each topic the author gives a concise but complete discussion of the basic equations, a clear account of the methods employed in their solution, and a summary of the results obtained. He also discusses the physical aspects of each topic in a clear and concise way. For the reader who is acquainted with modern methods of mathematical physics this article provides a useful introduction and guide to the rapidly expanding literature on magnetohydrodynamics. *D. Layzer* (Cambridge, Mass.).

De, J. The analogue of Kelvin's theorem in hydromagnetics. Naturwissenschaften 44 (1957), 256.

An analogue to Kelvin's theorem is proved for a perfectly conducting inviscid incompressible fluid, namely, that the circulation about a closed magnetic line is a constant of the motion. *A. A. Blank* (New York, N.Y.).

Jones, E. E. The magnetostatic characteristics of two non-magnetic elliptic cylinders. J. Math. Phys. 35 (1956), 266–277.

The magnetic field at all points inside and outside two equal parallel cylindrical conductors of elliptical cross-sections is determined when equal currents flow in the cylinders in the same or in the opposite directions. The inductive capacities of the cylinders are equal to that of free space. A series method of solution is used which follows that originated by Green [Quart. J. Math. Oxford Ser. 18 (1947), 167–177; MR 9, 113] for hydrodynamical problems. Expressions for the forces and couples acting on the cylinders are given and are evaluated numerically. The solutions of the problems when the inductive capacities of the cylinders are not equal to that of free space are indicated. *A. E. Green.*

Page, Chester H. Frequency conversion with nonlinear reactance. J. Res. Nat. Bur. Standards 58 (1957), 227–236.

A lossless nonlinear impedance subject to an almost periodic voltage (sum of sinusoids) will absorb power at some frequencies and supply power at other frequencies. Necessary and sufficient relations among these powers are found. It is shown that simple cubic capacitors ($Q\propto V^3$) are sufficient for producing any possible conservative modulation or distortion process.

Author's summary.

Edelmann, Hans. Über die Anwendung von Übertragermatrizen in Untersuchungen auf dem Netzmodell. Arch. Elek. Übertr. 11 (1957), 149–158.

The transformation matrix, associated with the method of symmetrical components, is physically realized by a network of ideal transformers. *G. Kron.*

Mayer, Daniel. Die Behandlung der elektromechanischen Übergangserscheinungen in der Theorie der elektrischen Maschinen mittels Tensorrechnung. Arch. Elektrotech. 42 (1956), 331–350.

The nonholonomic forms of the dynamical equations of Lagrange are developed for the transient and small oscillation analysis of the hypothetical "generalized" rotating electrical machine. From the latter the equations of all industrially useful machines can be derived by routine tensorial transformations. The theory is illustrated with the solution of the small oscillation characteristics of a direct-current shunt generator and of an alternating-current repulsion motor. *G. Kron.*

Lovass-Nagy, V.; and Szendy, Ch. Application of the matrix calculus to the investigation of transformers in arbitrary connection. Acta Tech. Acad. Sci. Hungar. 16 (1957), 311–352. (German, French, and Russian summaries)

The steady-state performance of n-phase, balanced, multiwinding transformers is treated by tensorial methods. *G. Kron* (Schenectady, N.Y.).

Elldin, Anders. On equations of state for a two-stage link system. Ericsson Technics 12 (1956), 61–104.

The author tries to give an approximation to the probability of loss in a two-stage link system, in which he introduces the correlation between the stages. He does not succeed in getting better results than known before.

A. Jensen (Copenhagen).

Pötzl, Hans. Die minimale Rauschzahl von nicht angepassten Verstärkern. Arch. Elek. Übertr. 11 (1957), 177–181.

Gewinn und minimale Rauschzahl von Verstärkern, die sich durch eine Streumatrix darstellen lassen, werden für beliebige Betriebszustände (Anpassungsverhältnisse) berechnet. Die Bedingungen für die Darstellbarkeit eines Verstärkers durch eine Streumatrix werden angegeben.

Zusammenfassung des Autors.

Strandberg, M. W. P. Inherent noise of quantum-mechanical amplifiers. Phys. Rev. (2) 106 (1957), 617–620.

See also: Bennett, p. 938; Denison and Taylor, p. 939; Schlitt, p. 940; Fodor and Temes, p. 940; Rose, p. 971; Halfin, p. 974; Alzermann, p. 981.

Thermodynamics and Heat

Gasanov, R. G. The problem of cooling an infinitely long cylinder in a stratified medium. Akad. Nauk Azerbaĭdžan. SSR. Trudy Inst. Fiz. Mat. 3 (1948), 53–56. (Russian. Azerbaijanian summary)

The problem considered is the following: find a solution of the equation

$$(1)\qquad m(P)\frac{\partial T}{\partial t}(P,t)=\operatorname*{div}_P\,(K(P)\operatorname*{grad}_P T(P,t))$$

subject to the initial condition

$$(2)\qquad T(P,0)=f(P),$$

where $f(P)$ is such that $f(\infty)=\text{const}$.

The problem is converted into an integro-differential equation and the method of Laplace transforms used to obtain an equation for the transform:

$$(3)\quad T^*(Q,s)=\frac{1}{4\pi K(Q)}\iiint_\infty \omega(M,Q)m(M)f(M)\,dv-\frac{s}{4\pi}\iiint_\infty\frac{\omega(M,Q)m(M)}{K(Q)}T^*(M,s)\,dv,$$

where

$$T^*(Q,s)=\int_0^\infty e^{-st}T(Q,t)dt\ (\operatorname{Re} s>0).$$

C. G. Maple (Ames, Iowa).

Fletcher, G. C. The thermal expansion of solids. Phil. Mag. (8) 2 (1957), 639–648.

The assumptions involved in two formulae due to Grüneisen [Handbuch der Physik, Bd. X, Springer, Berlin, 1926, pp. 1–59] are investigated and used to obtain an explicit formula for the volume of a solid as a function of temperature. (From the author's summary.)

D. R. Bland (Manchester).

Pipes, Louis A. Matrix analysis of heat transfer problems. J. Franklin Inst. 263 (1957), 195–206.

Matrix methods used for resistance-capacity transmission lines can be applied directly to one dimensional heat conduction problems as governing differential equations are identical. The paper will be of value to engineers concerned with heat transfer problems. *D. R. Bland.*

See also: Schlitt, p. 940; Norbury, p. 963; Levin, p. 969; Boldizsár, p. 977.

Quantum Mechanics

★**Rose, M. E. Elementary theory of angular momentum.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London. x+248 pp. $10.00.

The theory of angular momentum is basically an application of the theory of the rotation group to quantum mechanics. In this form it has been known to physicists since Wigner's classic on the subject. Since that time, however, partly stimulated by improved experimental techniques, this branch of quantum mechanics has been developed considerably, especially through the basic work by Racah. By its nature, the application of the theory of angular momentum cuts across the boundaries of many diverse fields of physics, from the nuclear shell model, angular correlations, and atomic spectra, to the many branches of molecular spectroscopy. Nevertheless, no account of the general underlying theory has been available in book form since Wigner's work.
The thin booklet 'Notes on the quantum theory of angular momentum' [Addison-Wesley, Cambridge, 1953] by Feenberg and Pake is no exception, since it does not essentially exceed the material of standard quantum mechanics texts.

The present book is therefore extremely welcome and fills a definite need. In addition, it serves a double purpose; if offers many important results for the first time in book form, and it also makes this whole field accessible to a much larger audience. In this respect the author certainly chose the right approach; he skillfully succeeded in avoiding any explicit use of group theoretical knowledge. It is a fact that most physicists still prefer it that way.

The book is lucidly written, easy to follow, and demands only a minimum of prerequisite knowledge. Physical examples are presented throughout. No attempt at complete generality or formal conciseness was made, which makes it excellent reading for the physicist who wants to obtain a working knowledge of the underlying mathematical structure.

The twelve chapters are equally divided between general theory and applications. The former starts from an elementary review of the basic principles of quantum mechanics and leads to the Wigner-Eckart theorem, tensor operators, and the Racah coefficients. A mathematical appendix supplements the theoretical part. The applications cover a variety of problems, but are taken mainly from nuclear physics, the author's own speciality. Thus, e.g. applications to atomic spectroscopy, which after all is the basic source of our knowledge of angular momentum in quantum mechanics, are treated a little too lightly. As a whole, however, this book is a fine introduction to the problems of angular momentum, and there is little doubt that it will be used widely as an easy reference and as a clear text. *F. Rohrlich.*

Kompaneets, A. S.; and Pavlovskii, E. S. The self-consistent field equations in an atom. Soviet Physics. JETP 4 (1957), 328–336.

Following a method first used by Dirac [Proc. Cambridge Philos. Soc. 26 (1930), 376–385] an effective Hamiltonian is derived from the Hartree-Fock self-consistent field equations, which commutes with the density matrix. A suitable Fourier representation of this vanishing commutator is expanded such that the zeroth approximation is the quasi-classical limit which gives exactly the Thomas-Fermi equation. To the next approximation, the first correction (of order $Z^{-2/3}$) is found to satisfy a linear inhomogeneous second order differential equation and contains the well-known exchange correction first found by Dirac (loc. cit.) as well as an additional small "quantum correction". On the basis of this result other methods of improving the Thomas-Fermi approximation are criticized. *F. Rohrlich* (Iowa City, Iowa).

Kudriavtsev, V. S. On quasiclassical quantization. Soviet Physics. JETP 4 (1957), 527–530.

A method of calculating the energy levels in quasiclassical quantization is presented for the one-dimensional case. The value of the levels is obtained in the form of an expansion in $\hbar$. Under certain assumptions on the form of the potential energy $U(x)$, this expansion can be ob-

tained in a general form. Computations are carried out for a potential energy having a minimum and rising on either side of the minimum, i.e., of an oscillator type.
Author's summary.

Amat, Gilbert. Interactions de résonance vibrationnelles et rotationnelles. Cahiers de Phys. no. **77** (1957), 25–39.

This is a review article on certain aspects of the rotational and vibrational motion in a molecule. First the calculation of the vibrational-rotational energy of a molecule is outlined. Following this, we have a discussion of the perturbation method, with special regard to the phenomenon of quasi-degeneracy and resonances. Then the interaction of vibrational and rotational resonances is treated, including resonances of the first and second order, and higher order resonances are also mentioned. As an example, the theory of the CO_2 molecule is discussed from this point of view, and it is shown that the theory gives satisfactory agreement with the experimental data. *M. J. Moravcsik* (Upton, N.Y.).

Narumi, H. On the problem of energy eigenvalue and degeneracy in quantum mechanics. Phil. Mag. (7) **46** (1955), 293–294.

On the basis of the representations of the semi-simple Lie group generated by integral operators suitably deduced from classical equations of motions, it is possible to set an eigenvalue problem which in principle solves the problem of quantization for any system, provided it is possible to find the corresponding "Schroedinger group" for the system. This statement is not proved, but applied to the problem of the nonrelativistic hydrogen atom.
D. Rivier (Lausanne).

★**Thirring, Walter. Einführung in die Quantenelektrodynamik.** Franz Deuticke, Wien, 1955. xii+122 pp. DM 17.50.

The usual introductions into an advanced field like quantum electrodynamics follow one of two lines: Either they present the elementary part of the formalism in painful detail, or they concentrate on the applications using recipes instead of basic theoretical understanding. This slim book follows neither method, but attempts to convey to the reader a feeling and understanding of the subject matter by pointing out some of its most characteristic and interesting aspects. In this attempt the author is very successful as far as he goes. The main criticism of the book is perhaps that he does not go far enough.

Anyone with the prerequisite knowledge of quantum mechanics and the special theory of relativity should have no difficulty following the presentation. The notation is carefully explained and a very desirable order of magnitude estimate of the phenomena involved is given. A condensed but clear discussion of classical electrodynamics introduces the three parts of the book which constitute the main subject matter. As is desirable in an introduction of this kind, the emphasis is on physical understanding rather than on formal or mathematical rigor, although the reviewer feels that some of the mathematical difficulties might have been put into sharper focus. Part I concerns the free fields. After a more or less standard presentation of the general framework via the action principle, the free electromagnetic and electron fields are discussed. The author then puts considerable emphasis on the fluctuation phenomena and the quantum field theoretical aspects of measurability. Most introductory texts barely mention these characteristic features of the theory. Similarly, part II, entitled "fields in interaction", after a short general discussion devotes substantial sections to multiple photon emission, radiation reaction, and vacuum polarization. These topics, which were all essentially developed before the war, are typical of quantum electrodynamics and basic for its understanding.

The developments of the post-war period, which deepened so much our understanding of the theory and to which the author himself has made important contributions, unfortunately are not treated in the same vein. Rather, they are only briefly summarized as part III, less than ten percent of the book being devoted to them. This seems out of proportion compared to some of the details of the basic theory in part I. Despite the excellent experimental agreement of the theory and the author's promise (in the preface) to present a "collection of satisfactory and interesting results", applications are treated in a somewhat cursory fashion. An appendix and a useful set of problems with their solutions supplement and conclude the book. There is no author or subject index. It is well printed, but the figures make a poor appearance.

In summary we can say that the reader's desire for a fuller treatment of the later sections in the book is entirely due to the successful presentation of the first two parts which the novice will read with profit and the expert with enjoyment. *F. Rohrlich* (Iowa City, Iowa).

Symanzik, K. On scattering at very high energies. Nuovo Cimento (10) **5** (1957), 659–665.

A proof is presented that under certain assumptions the high energy limit of the total scattering cross section for the collision of two scalar particles must increase less strongly than proportional to the energy in the laboratory system (i.e. the system in which one particle is initially at rest). The underlying assumptions are the unitarity conditions and the general dispersion relations as derived from quantum field theory, but the crucial assumption is that the "tangent approximation" to the scattering amplitude also fulfills the dispersion relations. This approximation is a semi-classical high energy approximation in which summations over angular momenta are replaced by integrals over impact parameters, etc. This "impact parameter method" was used previously in conjunction with the dispersion relations [H. A. Bethe and F. Rohrlich, Phys. Rev. (2) **86** (1952), 10–16]. The author's result is equivalent to the statement that no "extra denominators" are necessary in the dispersion relation. The method is finally applied to pseudoscalar symmetric meson theory and the restrictions on the asymptotic behavior of the amplitudes are discussed.
F. Rohrlich (Iowa City, Iowa).

McConnell, J. Theory of the negative proton. Nuclear Phys. **1** (1956), 202–228.

The calculations presented in this paper are mainly a summary of the author's papers and related work on negative protons since 1944. Production cross sections for negative protons are computed in *ps* and *pv* coupling for *ps* mesons. Meson-nucleon as well as nucleon-nucleon collisions are considered. The methods used are a combination of the theory of radiation damping and the Weizsäcker-Williams method. The life-time of the negative proton is also estimated. Application to cosmic

radiation leads to the conclusion that these particles would be difficult to detect, unless produced artificially.
F. Rohrlich (Iowa City, Iowa).

Claesson, Arne. Application of causality arguments to Delbrück scattering. Kungl. Fysiogr. Sällsk. i Lund Förh. **27** (1957), no. 1, 9 pp.

For the calculation of zero-angle Delbrück scattering the dispersion relations offer very great simplifications over the conventional formalism. The author casts the problem into a form related to the dispersion relations but permitting (at least in principle) the computation of the complete angular distribution. This is accomplished by the exclusive use of retarded Green's functions S_R and D_R rather than the causal functions. They are introduced by the expansion of the field operators in powers of e and lead to an S-matrix element for Delbrück scattering whose integrand is, apart from trivial factors, of the form

$$F_{\nu\mu}(p)-\frac{i}{\pi}P\int\frac{d\tau}{\tau}F_{\nu\mu}(p+in\tau),$$

where p is the integration variable and n the time-like unit vector. This neat separation into absorptive and dispersive part permits one to compute the latter from the former. It is the hope of the author that the problem is thereby considerably simplified. *F. Rohrlich.*

Kastler, Daniel. Discussion mathématique de l'équation de Dirac. Ann. Univ. Sarav. **5** (1956), 153–185 (1957).

The theory of the Dirac equation is constructed in an abstract, purely algebraic way, with a self-contained presentation of the necessary mathematical concepts and theorems. The first main step is the construction of the Clifford algebra on a given complex vector space with a symmetric quadratic form (metric). This construction follows an unusual and elegant pattern, based on a realization of the Clifford algebra as a set of operators in the Grassmann algebra on the vector space. The spinor space is defined in terms of the (essentially unique) irreducible representation of the Clifford algebra. The next step concerns the spinor representations of the Lorentz group belonging to the metric vector space. Then follows the construction of the Dirac equation. Both points are treated essentially along familiar lines. *L. Van Hove.*

Kastler, Daniel. La quantification du champ de Maxwell en tant qu'algèbre symétrique sur un espace de Hilbert. Ann. Univ. Sarav. **5** (1956), 186–203 (1957).

The general formulation of second quantization presented earlier by the author [same Ann. **4** (1955), 206–237; MR **18**, 444] is applied to the electromagnetic field. One starts from a complex vector field with the natural definition for a Lorentz-invariant hermitian scalar product (this metric is not positive definite). One imposes the d'Alembert equation for mass zero and the Lorentz condition. One obtains in this way the Hilbert space of free photons with the Bleuler-Gupta metric. The latter appears very naturally, as does the gauge group, an advantage of the author's presentation. Consideration of symmetric tensors on the Hilbert space gives the quantized field theory. The electromagnetic potentials and the momentum-energy operators are calculated.
L. Van Hove (Utrecht).

Kastler, Daniel. La quantification du champ de Dirac en tant qu'algèbre de Grassmann sur un espace de Hilbert. Ann. Univ. Sarav. **5** (1956), 204–227 (1957).

The program carried out in the paper reviewed above for the Maxwell field is here performed for the Dirac field. Use is made of the formalism developed in the paper reviewed second above for the one particle Dirac theory.
L. Van Hove (Utrecht).

Dolginov, A. Z. Relativistic spherical functions. Soviet Physics. JETP **3** (1956), 589–596.

If one describes the nonrelativistic bound two-particle system one uses as basis functions the irreducible representations of the three-dimensional rotation group and then expands the wave function in terms of these with the help of the Clebsch-Gordan coefficients. This same procedure is worked out in the present paper in four dimensions. The eigenstates of the four-dimensional angular momentum operator are derived, which form a basis for the irreducible representations of the Lorentz group. The analogues of the Clebsch-Gordan coefficients are also given. The procedure is applied to the Bethe-Salpeter equation to separate the variables for two scalar particles. It is promised that this last point will be elaborated upon in a future paper. *M. J. Moravcsik* (Upton, N.Y.).

Borgardt, A. A. Meson field theory. III. Conservation of physical quantities. Soviet Physics. JETP **3** (1956), 312–314.

This is a translation by M. Hamermesh of the original Russian article [Z. Eksper. Teoret. Fiz. **30** (1956), 330–333; supplement to 30, no. 2, 6; MR **18**, 96].

Montaldi, E.; and Pusterla, M. Electron scattering in nuclear field with pair-creation. Nuovo Cimento (10) **5** (1957), 961–972.

Differential cross-section for title problem calculated by Feynman's graphs.

Sugie, A.; Hodgson P. E.; and Robertson, H. H. The contribution of tensor forces to $n-\alpha$ scattering. Proc. Phys. Soc. Sect. A. **70** (1957), 1–16.

The splitting of the p-phase shifts due to tensor forces is calculated for $n-\alpha$ scattering using the resonating group method and Gaussian wave functions and potentials. It is found that the tensor force can account for about 30% of the observed splitting. *Author's summary.*

Zwanzig, Robert W. Transition from quantum to "classical" partition function. Phys. Rev. (2) **106** (1957), 13–15.

The quantum-mechanical partition function for a system of interacting electrons and nuclei is examined in the "classical" limit, in which $\hbar\to 0$ in the nuclear kinetic energy operator while $\hbar$ is constant in the electronic kinetic energy operator. It is shown that in the "classical" limit, the apparent nuclear potential energy which appears in the partition function is actually the free energy of the electrons in a system of fixed nuclei, as a function of nuclear configuration and temperature. The lowest order quantum correction is obtained. The effect of the adiabatic approximation is studied; it leads to the correct "classical" limit but a formally inexact lowest order quantum correction. *Author's summary.*

De Tollis, B.; and Liotta, R. S. Interference in the double Compton effect. Nuovo Cimento (10) **5** (1957), 947–954.

The differential cross-section by double Compton effect is calculated in the case of identity of the two emitted photons. It results in a factor of interference $1/(4\pi)^2$ besides the factor 1/137 due to the perturbative development. *Author's summary.*

Ceolin, C.; and Taffara, L. On the scattering of K^+-mesons by nucleons in perturbation theory. Nuovo Cimento (10) 5 (1957), 435–447.

Using the method of Feynman and considering both the Λ and Σ-particles as intermediate states, the differential and total cross-sections for the scattering of scalar and pseudoscalar K^+-mesons by nucleons have been calculated in the second order approximation of perturbation theory. As a result, the various values are given which may be assumed by the ratios:

$$R=\frac{\text{process with charge exchange}}{\text{process without charge exchange}}\text{ and }R'=\frac{\sigma(K^+N)}{\sigma(K^+P)}$$

as a function of the ratio $s=g_\Lambda{}^2/g_\Sigma{}^2$ of the interaction constants for the reactions $N\rightarrow K+\Lambda$ and $N\rightarrow K+\Sigma$. These results are then discussed in relation to the few experimental data so far available. *Authors' summary.*

Fukutome, Hideo. Low's scattering equation and S-matrix. Progr. Theoret. Phys. 17 (1957), 383–400.

Fujita, Jun-ichi; and Miyazawa, Hironari. Pion theory of three-body forces. Progr. Theoret. Phys. 17 (1957), 360–365.

Arima, Akito; Horie, Hisashi; and Sano, Mitsuo. The l-forbidden magnetic dipole transitions. Progr. Theoret. Phys. 17 (1957), 567–580.

Konuma, M.; and Umezawa H. High energy behaviour of renormalizable fields. II. Nuovo Cimento (10) 4 (1956), 1461–1472.

Bogoliubov, N. N.; and Shirkov, D. V. The multiplicative renormalization group in the quantum theory of fields. Soviet Physics. JETP 3 (1956), 57–64.

The behavior of Green's function when large momenta are present presents a basic problem in the quantum theory of fields. When the series of perturbation theories are used for the representation of Green's function, with the coefficients being the sums of an infinite number of Feynman diagrams of various order, some deficiencies of this procedure may appear: restricted domains of convergence, etc. But such problems can be solved without any summations of infinite series, using the renormalizing group of quantum electrodynamics. This was noted by Stueckelberg and Petermann [Helv. Phys. Acta 26 (1953), 499–520] and was applied by Gell-Mann and Low [Phys. Rev. (2) 95 (1954), 1300–1312; MR 16, 315] to investigate the behavior of Green's function in the asymptotic region of large momenta. Bogoliubov and Shirkov show how to use results of ordinary perturbation theory not only in the case of large momenta but also in the region of infrared catastrophes. They start from expanding the scattering matrix in powers of the interaction parameter in such a way that no term contains divergencies, and obtain as fundamental condition the condition of transversality. It is pointed out that Gell-Mann and Low used the expression for the Green's function of a photon which is not transverse, thus obtaining a contradiction. Using a special set of transformations Bogoliubov and Shirkov derive Lee's differential equations for the multiplicative normalization group in the quantum theory of fields. As an example of the application of these equations, they derive the Green's function of spinor electrodynamics in the regions of infrared catastrophes. The method used does not require summation over infinite systems of Feynman diagrams. *M. Z. Krzywoblocki.*

Halfin, L. A. The condition of causality and the criterion of physical realizability in quantum field theory. Dokl. Akad. Nauk SSSR (N.S.) 111 (1956), 345–347. (Russian)

The author calls attention to the Theorem XII of R. E. A. C. Paley and Norbert Wiener "Fourier transforms in the complex domain" [Amer. Math. Soc. Colloq. Publ., v. 19, New York, 1934], which gives a necessary and sufficient condition in terms of its Fourier transform, for a function to vanish on the half line. The Fourier transforms of such functions are said to correspond to "physically realizable" electrical filters in network theory [see G. E. Valley, Jr., and H. Wallman, Vacuum tube amplifiers, McGraw Hill, New York, 1948, Appendix A] inasmuch as they guarantee that the "causality condition" be satisfied, i.e. that the response to an impulse function (which is zero for $t<0$) shall also be zero for $t<0$. Most previous physical discussions have involved the Hilbert transform relations between real and imaginary parts of analytic functions on the real or imaginary line (Kronig-Kramers relation.) The Paley-Wiener theorem involves only the modulus of the analytic function. The author remarks that the use of the theorem should restrict the physically admissable energy dependence of scattering cross-sections at high energies. *D. Falkoff.*

Higgs, P. W. Vacuum expectation values as sums over histories. Nuovo Cimento (10) 4 (1956), 1262–1273.

This paper examines some of the limiting processes, which were treated in a cavalier manner by Matthews and Salam [Nuovo Cimento (10) 2 (1955), 120–134; MR 17, 693] in their evaluation of field theory propagators by functional integration. The result is a justification in detail of the procedure employed for the non-interacting Bose field. *P. T. Matthews* (Rochester, N.Y.).

Peierls, R. E.; and Yoccoz, J. The collective model of nuclear motion. Proc. Phys. Soc. Sect. A. 70 (1957), 381–387.

The problem of introducing coordinates suitable for the description of collective motions of a system of particles is encountered in many places in physical theory. It has received special emphasis recently, owing to the success of the collective model of nuclear motions. The present paper is concerned particularly with the nuclear problem, and the relationship existing between the collective and shell models of the nucleus. The general procedure is to take a set of degenerate, or nearly degenerate, wave functions from the shell model, and to write a more complete wave function as a linear combination of this set. The composite wave function is then subjected to the usual variation principle to determine the best values of the coefficients. The coefficients, which may form a discrete or continuous set, play the role of the collective coordinates. The theory gives an extension of the paper by J. P. Elliot and T. H. R. Skyrme [Proc. Roy. Soc. London Ser. 232 (1955), 561–566]. *E. L. Hill.*

Suzuki, Ryozi. Deuteron photodisintegration at high energies. Progr. Theoret. Phys. 15 (1956), 536–544.

The author investigates a particular meson effect in the photodisintegration of the nucleus. The term is one in which the incident photon is absorbed by the proton; a meson is emitted and finally absorbed by the proton. The calculation assumes that the meson-nucleon interaction is strongest in the isotopic spin 3/2, angular momentum 3/2 state of the meson-nucleon system. A single

level resonance formula is used to represent the effect of this interaction. Comparison with experiment is good in the 250 Mev region, but there is a large discrepancy at 150 Mev. There is no discussion of the effect of Siegert's theorem. *H. Feshbach* (Cambridge, Mass.).

Kahan, T.; Rideau, G.; et Roussopoulos, P. Les méthodes d'approximation variationnelles dans la théorie des collisions atomiques et dans la physique des piles nucléaires. Mémor. Sci. Math., no. **134**. Gauthier-Villars, Paris, 1956. 82 pp. 1200 francs.

This is a review article in which a particular type of variational principle for scattering is developed and applied to various cases. The general method involves two steps: (1) the formulation of the problem including boundary conditions in terms of an integral equation and (2) the development of an expression whose stationary value upon variation of a trial function yields the integral equation. The method is applied to formulate variational expressions for problems involving scalar waves, in electromagnetic theory, in quantum-mechanical scattering, and to one group stationary state Boltzmann equation. This last has, of course, application to the theory of the reactor. A summary of the contents of the paper follows.

In Chapter I, scattering is considered. Definitions of cross sections are given, and the problem of scattering by a central field set up. The integral equation for this case is derived and the Born approximation given. Phase shift analysis is also discussed. Chapter II opens with a general discussion of a variational principle for integral equations. A second part of Chapter II is devoted to a discussion of scattering from the point of view of time-dependent theory. The scattering transition matrix and reactance matrix are introduced, their integral equations and consequent variational principles derived. A variational principle for the phase shifts is also found. In Chapter III scattering from a complex system is discussed and, in particular, the impulse approximation derived. Chapters IV and V give various applications. *H. Feshbach.*

Butler, S. T. Direct nuclear reactions. Phys. Rev. (2) **106** (1957), 272–286.

The mechanism investigated is the direct interaction between an incident particle and a particle in the target nucleus. This can lead to a variety of reactions such as the pickup reaction (p, d), inelastic proton or alpha particle scattering, the (n, p) reaction and so on. The author develops the theory for these reactions as they occur via the direct interaction by essentially the same formalism in which the direct interaction is considered to be a perturbation, the formation of the compound nucleus being dominant. Explicit expressions for cross sections for these processes is obtained by making the following approximations, the justification of which the author discusses: (1) the interaction takes place in the "surface" of the target nucleus and (2) plane waves for the incident and emergent particles. *H. Feshbach.*

Saperstein, A. M.; and Durand, Loyal, III. Boundary value treatment of nucleon-nucleon phase shifts. Phys. Rev. (2) **104** (1956), 1102–1113.

An attempt by Feshbach and Lomon [Phys. Rev. (2) **102** (1956), 891–904] to fit the nucleon-nucleon scattering data up to 274 Mev laboratory energy by the use of energy independent boundary conditions is examined critically. It is found that there were discrepancies when comparisons were made with experimental data on small angle p-p scattering, n-p and p-p polarization. These discrepancies could not be accounted for by permitting small changes in the boundary conditions. The authors attempt to obtain a better fit by letting the boundary conditions be energy-dependent. *H. Feshbach.*

Goldstone, J. Derivation of the Brueckner many-body theory. Proc. Roy. Soc. London. Ser. A. **239** (1957), 267–279.

In Brueckner's theory of many-body systems the saturation property of nuclear matter can be derived provided certain terms, the so-called "unlinked cluster" terms do not contribute. In this paper the author furnishes the first rigorous proof of the absence of these terms by using a novel formulation of the problem. A system of fermions in mutual interaction is considered. Its Hamiltonian is assumed to be of the form $H=H_0+H_1$ with $H_0=\sum_i (T_i+V_i)$, $H_1=\sum_{i<j} v_{ij}-\sum_i V_i$. If one assumes that the lowest eigenstate of H_0 is non-degenerate, and if one uses the formalism of second quantization, it is possible to exploit the Feynman diagram method of quantum electrodynamics to obtain a formal solution in perturbation expansion for the total energy of the system. This ingenious method is based on the observation that (a) any state of A particles is determined by specifying the excited one-particle states which are occupied, and the unexited one-particle states which are unoccupied, and that (b) there exists a correspondence with the theory of electrons in that excited occupied one-particle states correspond to negatons and unexcited unoccupied one-particle states correspond to positons. The ground state then corresponds to the vacuum. The eigenvalue of H for this ground state is $E_0+\Delta E$, where ΔE is given by the limit of a ratio [Gell-Mann and Low, Phys. Rev. (2) **84** (1951), 350–354; MR **13**, 413]. It is then easy to show that ΔE is given by a sum over connected diagrams only, thus proving the absence of "unlinked clusters". For completeness the Hartree-Fock and Brueckner methods are also derived with this diagram formalism. *F. Rohrlich.*

Landau, L. D. The theory of a Fermi liquid. Soviet Physics. JETP **3** (1957), 920–925.

Consider a Fermi gas at temperatures which are low in comparison with the temperature of degeneration. The collision probability for a given atom is proportional to the square of the temperature. Thus the indeterminacy of the momenta associated with the finite path length is small for low temperatures. The basic assumption of Landau's theory for the construction of spectrum is that in the gradual transition from the gas to the liquid, the classification of the levels remains invariant. The role of particles is assumed by the "elementary excitations", each possessing a definite momentum and obeying Fermi statistics. Their number coincides with the number of particles in a liquid. The energy of a particle depends on the state of surrounding particles, but the energy of the whole system is a functional of the distribution function. These assumptions allow the author to derive expressions for 1) the energy density of the system taking account of the fact that the particles possess spin, 2) the entropy, 3) heat capacity, and 4) the effective mass of the quasiparticles.

For a liquid which is not in an external field it follows from the principle of Galilean relativity that the momentum arriving at a unit volume must be equal to the density of mass flow. This enables one to find a relation

between the real and effective masses. Other formulae, derived by the author, refer to compressibility and magnetic susceptibility of the Fermi liquid, and the momentum and energy flow. *M. Z. Krzywoblocki.*

Fairbairn, W. M. The stripping theory of deuteron reactions and the inelastic scattering of deuterons. Proc. Roy. Soc. London. Ser. A. **238** (1957), 448–472.

The cross section of the stripping process of the deuteron reactions is calculated under the following approximations: (a) neglect of the deuteron-nucleus coulomb forces; (b) neglect of the proton-nucleus interaction; (c) neglect of the proton-neutron coupling in the scattered wave. If approximation (c) is relaxed the neutron-proton interaction gives rise to the inelastic scattering of deuterons. It is shown that the experimental data are consistent with this mechanism of scattering rather than with the picture of an intermediate compound nucleus consisting of the target nucleus plus the whole deuteron.
J. Leite Lopes (Pasadena, Calif.).

Das, T. P.; Roy, D. K.; and Roy, S. K. Ghosh. Quadrupolar nuclear spin-lattice relaxation in crystals with body-centered cubic lattice structure. Phys. Rev. (2) **104** (1956), 1568–1572.

The authors have calculated the relaxation rate of nuclear spins caused by quadrupolar intersections with the ions in a body-centered cubic ionic crystal, using standard methods first applied to this problem by Van Kranendonk [Physica **20** (1954), 781–800].
P. W. Anderson (Murray Hill, N.J.).

Maki, Ziro. On a theory of a composite model of elementary particles. Progr. Theoret. Phys. **16** (1956), 667–683.

Assuming a direct 4-Fermi interaction between nucleons, the possibility of obtaining π-mesons as composite states of nucleons-anti-nucleons is investigated. The covariant formalism of Salpeter and Bethe [Phys. Rev. (2) **84** (1951), 1232–1242; MR **14**, 707] with a cut-off at about nucleon-mass is employed. Following a suggestion of Sakata [Progr. Theoret. Phys. **16** (1956), 686–688] it is shown that k-mesons (considered as composite states of Λ and anti-nucleons) can be treated similarly. The mathematical structure resembles the work of Heisenberg, Kortel and Mitter [Z. Naturf. **103** (1955), 425–446; MR **17**, 330]. *A. Salam* (London).

Ritus, V. I. The scattering of photons by nucleons and nuclear isobars. Soviet Physics. JETP **3** (1957), 926–934.

Oehme, R. Causality and dispersion relations for the scattering of mesons by fixed nucleons. Nuovo Cimento (10) **4** (1956), 1316–1328.

van der Spuy, E. The interaction of neutron and alpha -particle. Nuclear Phys. **1** (1956), 381–414.

Alder, K.; Bohr, A.; Huus, T.; Mottelson, B.; and Winther, A. Study of nuclear structure by electromagnetic excitation with accelerated ions. Rev. Mod. Phys. **28** (1956), 432–542.

A survey article.

Sirlin, A. Spectrum of target Bremsstrahlung at small angles. Phys. Rev. (2) **106** (1957), 637–645.

Stuart, G. W. Effect of neutron interaction on criticality. II. J. Appl. Phys. **28** (1957), 677–679.

Oda, M. A calculation on the structure of the nucleonic cascade in the atmosphere. Nuovo Cimento (10) **5** (1957), 615–627.

Messel, H. On the solutions of the fluctuation problem in cascade showers. Nuovo Cimento (10) **4** (1956), 1339–1348.

Popov, Iu. A. Solution of the fundamental diffusion equation for cosmic ray particles emitted by a constant energy concentrated pulsed source. Soviet Physics. JETP **4** (1957), 85–90.

Gatto, R. Quantum numbers of the Lee-Yang parity doublet theory of strange particles. Z. Physik **147** (1957), 261–263.

Joos, H.; Leal Ferreira, J.; and Zimerman, A. H. On the kinematic properties of a system of two Dirac particles. An. Acad. Brasil. Ci. **28** (1956), 253–274.

The authors consider the relativistic Schroedinger equation for an electron and proton. The interaction between this contains a Coulomb attraction, the Breit term, and a term which arises from the anomalous magnetic moment of the proton. They make an exact reduction of the Schroedinger to radial equations by making use of the eigenfunctions of the total angular momentum operator and the inversion operator. They can, after considerable manipulation, reduce their equations to two coupled linear first-order differential equations. By taking suitable limits they can compare their results to the standard calculations in which corresponding approximations are inserted from the beginning in the Schroedinger equation. *H. Feshbach* (Cambridge, Mass.).

See also: Lowdenslager, p. 913; Strandberg, p. 970; Dugas, p. 982.

Relativity

Fourès-Bruhat, Yvonne. Sur l'intégration des équations de la relativité générale. J. Rational Mech. Anal. **5** (1956), 951–966.

On sait que le problème de l'intégration des équations de la relativité générale se subdivise en deus problèmes distincts: le problème de l'évolution dans le temps et le problème de la recherche des conditions initiales. Le premier de ces problèmes consiste en la détermination d'une solution des équations de champ correspondant à des données de Cauchy sur une variété initiale S ($x^4=0$), orientée dans l'espace. Le second consiste dans la détermination des données intiales qui doivent satisfaire sur S aux 4 équations $S_\alpha{}^4=0$ ($S_{\alpha\beta}$ tenseur d'Einstein, $\alpha, \beta=1, \cdots, 4$) dans le cas extérieur ou à 4 équations analogues reliant données initiales et distribution énergétique dans le cas intérieur. Le premier problème a été entièrement résolu, sous de simples hypothèses de différentiabilité, par l'A. [Acta Math. **88** (1952), 141–225; MR **14**, 756]. C'est au second problème abordé antérieurement par le rapporteur [J. Math. Pures Appl. (9) **23** (1944), 37–63; MR **7**, 266] que le présent travail est consacré.

L'A. écrit en repères orthonormés les 4 équations d'Ein-

stain à étudier, soit

$$\overline{\nabla}_h P - \overline{\nabla}_k P_{hk} = S_{4h}, \quad \bar{R} + H^2 - P^2 = -2S_{44},$$

où P_{hk} ($h, k = 1, 2, 3$) définit la seconde forme quadratique fondamentale de S (forme de plongement), P la courbure moyenne, H^2 le carré de tenseur P_{hk} et où les barres indiquent les éléments relatifs à S muni de la métrique $\overline{ds}^2$ induite par plongement. Supposant donnée sur S une métrique ds^{*2}, conforme à $\overline{ds}^2$, et les $P_{hk}=0$ par $h \neq k$, l'A. montre que les équations permettent la détermination des inconnus restantes, c'est-à-dire du facteur de conformité et des rayons de courbure. Le problème se trouve ramené à l'étude d'une équation elliptique et d'un système hyperbolique, quasi-linéaires et où les coefficients des dérivées d'ordre supérieur sont des fonctions données des variables. Un théorème d'existence et d'unicité est developpé pour des données à la frontière Σ d'un domaine Δ de S suffisamment petit. L'A. montre de plus que pour des rayons de courbure suffisamment grands sur Σ et pour $R^* \leq 0$ et de valeur absolue assez petite, on peut construire une solution globale du problème des conditions initiales. Dans le cas où S est minima ($P=0$), les résultats coincident grosso modo avec ceux du rapporteur.

A. Lichnerowicz (Paris).

Eisenhart, Luther P. A unified theory of general relativity of gravitation and electromagnetism. IV. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 333–336.

In this paper the author derives in a different manner the result of part III [same Proc. **42** (1956), 878–881; MR **18**, 543].

V. Hlavatý (Bloomington, Ind.).

Meister, H. J. Die Bewegungsgleichungen in der Theorie des Gravitationsfeldes mit einer Feldfunktion. Z. Physik **147** (1957), 531–543.

The author deduces the equations of motion of particles from the field equations in Papapetrou's theory of gravitation [Z. Physik **139** (1954), 518–532; MR **16**, 870], using the Fock-Papapetrou approximation method [see Papapetrou, Proc. Phys. Soc. Sect. A. **64** (1951), 57–75; MR **12**, 546] as far as the second approximation.

F. A. E. Pirani (London).

See also: Dugas, p. 982.

Astronomy

Kopal, Zdeněk; and Kurth, Rudolf. The relation between period and times of the maxima or minima of variable stars. Z. Astrophys. **42** (1957), 90–100.

The first-order difference equation

$$M(E+1)-M(E)=P(E)$$

relates the period P of a variable star at epoch E to the times of maximum (or minimum) light at successive epochs E and $E+1$. This equation has the well-known solution

$$M(E)-M(0)=\int_0^E P(E)dE+\tfrac{1}{2}\{P(0)-P(E)\} + \sum_{j=1}^{n} \frac{B_{2j}}{(2j)!}\{P^{(2j-1)}(E)-P^{(2j-1)}(0)\}+R_n,$$

where the B's are the Bernoulli numbers. The authors point out that the error involved in neglecting the terms following the integral — a practice commonly induiged in by astronomers — may be considerable under certain circumstances, and illustrate this point with examples.

D. Layzer (Cambridge, Mass.).

Longe, P.. Viscosité turbulente et stabilité vibrationnelle des étoiles. Bull. Soc. Roy. Sci. Liège **25** (1956), 541–553.

A stellar model with a large convective zone is vibrating with small radial oscillations. The effect of turbulent viscosity on motions of this type is considered, using the analogy of molecular viscosity. The coefficient of dynamical viscosity is a function of distance from the center of stellar model. It is shown that the viscosity produces a large total damping localized chiefly in the outer shells. There is also a considerable transfer of kinetic energy from the external layers towards the central regions. The relative amplitude of the oscillations is also made much more uniform throughout the model. For a sufficiently large convective zone the damping by turbulent viscosity might be more than compensated by the instability consequent on the enhanced nuclear reactions in the central regions. This enhancement would occur through the large variations in the central temperature brought about by the uniformisation of the oscillations. *G. C. McVittie.*

See also: Haselgrove and Hoyle, p. 939; McVittie, p. 949; Neymann and Scott, p. 949.

Geophysics

Boldizsár, T. The thermal field of the earth's crust and its influence on the ventilation of deep and hot mines. Acta Tech. Acad. Sci. Hungar. **16** (1957), 415–428. (German, French, and Russian summaries)

Matschinski, Matthias. Statistical method in geophysical prospecting. Ann. Geofis. **9** (1956), 151–165.

Maškovič, S. A. On prediction of atmospheric pressure by means of a high-speed electronic computer. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 245–248. (Russian)

After simplifying somewhat the system of four partial differential equations to which short range prediction for "flat" earth can be reduced the author used the BESM computer of the U.S.S.R. Academy of Sciences. In an example (illustrated by three maps) the complete time used for the computation for 24 hours in advance was about 40 minutes; the number of operations performed was about 20,000,000.

Ivakin, B. N. Similarity of elastic wave phenomena. I. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1956**, 1269–1281. (Russian)

In this paper are studied the problems related to the dynamic similarity in the propagation of seismic waves in two geometrically similar nonhomogeneous media consisting each of homogeneous, isotropic and ideally elastic parts. Physical criteria of such dynamic similarity are deduced.

E. Kogbetliantz (New York, N.Y.).

Adachi, Ryuzo. Fundamental relations on the seismic prospecting. Kumamoto J. Sci. Ser. A. **2** (1955), 253–258.

Elementary relations between the equations $y=g(x)$,

$y=f(x)$ of two cylindrical surfaces of separation in the case of three layers with velocities v_1, v_2, v_3, their derivative, $g'(x)$, $f'(x)$, the equation $t=\varphi(x)$ of time-curve etc. {On page 257 the transformation to polar coordinates of the differential equation (21) allows its integration (form 22) only when $X_0=0$ which is not mentioned at all.}

E. Kogbetliantz (New York, N.Y.).

Baranov, V. A new method for interpretation of aeromagnetic maps: pseudo-gravimetric anomalies. Geophysics **22** (1957), 359–382; discussion 382–383.

This paper by one of the leading European geophysicists is a very valuable and important contribution to the interpretation methods of magnetic and aeromagnetic surveys. In this new method the map of the total anomalous magnetic intensity T (or of the vertical component Z) is so transformed, that the resulting anomaly map (called by the author "pseudo-gravimetric") represents what the vertical magnetic intensity would be if the disturbing masses were magnetized vertically.

Thus, the distortions dues to the inclination of the normal magnetic field and to the orientation of the magnetized structure with respect to the magnetic meridian are eliminated. In other words, the effect of the transformation is such as if the magnetized masses were located just below the magnetic pole of the Earth.

Naturally, the correct interpretation of the transformed anomalies becomes easy because they are located on the vertical of the magnetized masses and do not depend on their orientation, nor on the inclination of the normal field. Baranov's method will be undoubtedly very useful in the interpretation of magnetic surveys, especially of those that are aeromagnetic.

Written with much elegance the paper is perfectly clear and easy to understand notwithstanding the very important part played in it by mathematical analysis. It is an excellent illustration of the scientific revolution caused by the introduction of electronic computers: Naranov's transformation necessitates the computation of a double integral for each one of thousands of grid-values on the transformed map. Detailed practical formulae are given for this computation which are well adapted to electronic computing equipment without the use of which the amount of these computations would be prohibitive.

The paper is also an excellent example of a harmonious combination of mathematical physics, mathematical analysis, and geophysics, the three sciences a combined knowledge of which is absolutely necessary for a prospecting geophysicist to be able to understand the true meaning of his surveys and to interpret them correctly.

E. Kogbetliantz (New York, N.Y.).

Diamantides, N. D.; and Horowitz, M. Autocorrelation of the earth's crust with analog computers. Rev. Sci. Instrum. **28** (1957), 353–360.

If h is the terrain height (datum mean sea level), t is any distance along the ground in a fixed direction from 0 to the full profile length T, and τ is the correlation interval, then the autocorrelogram $\Phi(\tau)$ of $h(t)$ is

$$\Phi(\tau)=\frac{1}{T}\int_0^T h(t)h(t+\tau)dt.$$

Data from this computation can be used to obtain mean terrain height, rms relief and rms slope.

The paper describes in detail a new method of computing $\Phi(\tau)$ with diagrams of the circuitry employed in the GEDA analog computer and results are given for terrain profiles of Ohio and Arizona. *P. D. Thomas.*

Lyapin, N. M. On a new trigonometric series for the radius of curvature of a normal section of the terrestrial ellipsoid. Rostov. Gos. Ped. Inst. Uč. Zap. no. **3** (1955), 89–94. (Russian)

If R denotes the radius of curvature, at a point M of the terrestrial ellipsoid, of a normal section of azimuth α, we have, by Euler's theorem:

$$1/R=\cos^2\alpha/\rho+\sin^2\alpha/p,$$

where ρ and p are, respectively, the radii of curvature of the meridional and latitudinal sections at N, and thus:

$$\rho=a(1-e^2)/(1-e^2\sin^2\varphi)^{3/2},\quad p=a/(1-e^2\sin^2\varphi)^{\frac{1}{2}},$$

φ being the latitude of M.

Starting with those three formulas the author obtains, as a result of a considerable computational effort, the following series:

$$R=(\rho p)^{\frac{1}{2}}$$
$$\left\{1-\tfrac{1}{2}\cos^2\varphi\sum_{k=1}^{n}e^{2k}[\cos 2\alpha+\sin^4\alpha(1-\sin^{2k-2}\varphi)]\right\}.$$

N. A. Court (Norman, Okla.).

Chovitz, Bernard H. Classification of map projections in terms of the metric tensor to the second order. Boll. Geodes. Sci. Affini **11** (1952), 379–394.

The classification of map projections with regard to their characteristics in the neighborhood of a point is developed by examining the second order aspects of the metric tensor for various geodesic coordinates on the surface to be projected. In terms of geodesic coordinates, u^1 and u^2, which provide the projection on the plane, the metric tensor may be expanded as $g_{ij}=\sum a_{ij}^{rs}(u^1)^r(u^2)^s$. Since the coordinates are geodesic, $a_{ij}^{rs}=0$ $(r+s=1)$. Furthermore the coefficients, a_{ij}, describe only the behavior at the origin of coordinates. The second order coefficients, a_{ij}^{rs} $(r+s=2)$, provide the initial basis for distinguishing projections. Basic characteristics such as conformality, equivalence, equidistance, etc. are shown to imply conditions on these second order coefficients. A tabulation is provided classifying all conventional projections in terms of numerical values for these same coefficients. *N. A. Hall* (New Haven, Conn.).

Chovitz, Bernard H. Some applications of the classification of map projections in terms of the metric tensor to the second order. Boll. Geodes. Sci. Affini **13** (1954), 47–67. (English and Italian)

The methods developed in the paper reviewed above of expressing characteristics of map projections in terms of the second order expansion coefficients, a_{ij}^{rs} $(r+s=2)$ of the metric tensor are applied to the determination of the specific items: maximum scale error, direction of maximum scale error, angle error, maximum angle error, and transformation between geographic and non-geographic coordinates. *N. A. Hall* (New Haven, Conn.).

Chovitz, Bernard H. A general formula for ellipsoid-to-ellipsoid mapping. Boll. Geodes. Sci. Affini **15** (1956), 1–20. (English and Italian)

The paper reviewed above develops general formulae correct to the third order in geographic coordinates for mapping between an ellipsoid and the plane. These results are extended to the mapping between one ellipsoid and another. Consideration is given both to the changes in the

ellipsoid and to the varieties of projection systems which may be used. Extensive comparison with similar results published elsewhere is made. *N. A. Hall.*

Wunderlich, W. Zur rechnerischen Durchführung des Vierpunktverfahrens. Österreich. Z. Vermessgswes. **45** (1957), 9–13.

★ **Danilow, W. W. Präzisionspolygonometrie.** VEB Verlag Technik, Berlin, 1957. 228 pp.

This is the most comprehensive exposition of the traverse method of geodetic survey published up to now. The author's principal thesis is that traverse can be employed as precisely as triangulation in order to obtain 1st, 2nd and 3rd order survey work. The mathematical portions of the book are Chapter II, on the determination of errors in a traverse loop, and Chapters X and XI, on the adjustment of traverse loops. The remainder of the work is concerned with such topics as instrumentation and methods of observation in the field.

The discussion of the determination of error (more precisely, the root mean square error) covers the computation of the orientation, radial, transverse, and coordinate errors in a loop under all possible circumstances—free, attached at one or both ends, and with and without azimuthal control. Also derived are formulas for the permissible curvature of a loop.

The chapters on the adjustment of traverse loops present this topic on a level of mathematical rigor comparable to similar material on triangulation adjustment. The following topics are discussed thoroughly and with illustrative examples: Computation of the length of the main sides of a loop, adjustment of a loop of arbitrary form fixed at both ends, inclusion of azimuthal conditions, adjustment of a 1st order loop in geographic coordinates, adjustment with conditioned observations, adjustment of a loop in connection with adjoining triangulation, and methods of approximate adjustment.

B. Chovitz (Washington, D.C.).

See also: Olivier, p. 968.

OTHER APPLICATIONS

Games, Economics

★ **Blackwell, David. Controlled random walks.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 336–338. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

The author restates a major result of a previous paper of his [Pacific J. Math. **6** (1956), 1–8; MR **18**, 450] and applies it to deduce the following result of Hannan and Gaddum: If $A=\|a_{ij}\|$ is the payoff matrix for a zero-sum two person game and player II uses a mixed strategy q, then without prior knowledge of q, player I can play the game in such a manner that as the number of plays increases, I's average expected income will approach

$$h(q)=\max \sum a_{ij}q_j.$$

The author repeats some examples relating to extensions of his theorem and points out some open questions also related to it. *H. M. Gurk* (Philadelphia, Pa.).

Morishima, Michio. An analysis of the capitalist process of reproduction. Metroecon. **8** (1956), 171–185.

The author shows that one version of an n-sector Marxian scheme of capital accumulation can be formulated as follows. Let $Y(t)$ be the vector of values of output in the n departments at time t, and let $D^{(1)}(t)$ be the vector of demands generated by $Y(t)$. Then $D^{(1)}(t)=HY(t)$ where H is a square matrix compounded out of such standard Marxian parameters as rates of surplus value, organic compositions of capital, etc., all assumed constant. H has non-negative elements and unit column sums, and could double as the transition matrix of a (primitive) finite Markov chain. In general $D^{(1)}\neq Y$, so that excess demands and supplies appear. The author supposes that within each period a process of successive approximation goes on by means of a kind of Edgeworthian recontract which generates further demand vectors $D^{(u+1)}(t)=D^{(u)}(t)+H[D^{(u)}(t)-D^{(u-1)}(t)]=HD^{(u)}(t)$. Demands are said to have degree of flexibility u if the process terminates at the uth stage and $Y(t+1)=GD^{(u)}(t)$ where G is the matrix $[\delta_{ij}(1+r_i)]$ and the r's are non-negative. Hence $Y(t+1)=GH^uY(t)$. The cases $u=0$, $u>0$, and $u\to\infty$ are separately considered. In the second case, for example, since the characteristic roots of GH^u are those of H^uG, the dominant solution grows at a rate $\lambda_1-1=\sum r_iq_i$, where q_i are the components of the eigenvector associated with the Frobenius root λ_1. In the case $u\to\infty$, the well-known properties of Markov chains show that the system moves instantaneously into a state of balanced exponential growth. There are some remarks about the decomposable case. *R. Solow.*

Conterno, Cesare. Sull'errore dell'interpolazione lineare per alcune funzioni della matematica finanziaria ed attuariale. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **90** (1955–56), 273–283.

Kometani, Eizi; and Kato, Akira. On the theoretical capacity of an off-street parking space. Mem. Fac. Engrg. Kyoto Univ. **18** (1956), 315–328.

This paper deals with a stochastic model for the occupation of parking places in a system of parking lots. Although there is a number of potentially interesting models that one could study, the reviewer was unable to determine from this paper exactly what the authors were trying to do or what model they were proposing. The capacity of a lot, however, was defined by selecting a tolerable limit for the probability of having a full lot. The authors believe "that this solution will provide rational and scientific basis for the parking projects in future". *G. Newell* (Providence, R.I.).

Kimball, George E. Some industrial applications of military operations research methods. Operations Res. **5** (1957), 201–204.

Beckmann, Martin J.; and Laderman, Jack. A bound on the use of inefficient indivisible units. Naval Res. Logist. Quart. **3** (1956), 245–252 (1957).

Solving a discrete-variable linear programming problem often requires enumeration of its entire domain. The authors find a simple upper bound for those values of x_j which may occur in the solution of the problem: Minimize $\sum_{j=1}^n s_jc_jx_j$ subject to $\sum_{j=1}^n s_jx_j\geq p$, x_j non-negative and

integral, (s_j, c_j, p) non-negative constants with $c_1 < \cdots < c_n$, $s_1c_1 > \cdots > s_nc_n$. The result is given in detail for $n=2$ and interpreted as an allocation problem. *P. Wolfe.*

Salvadori, Mario. La programmazione lineare. Civiltà delle Macchine **5** (1957), 22–24.
Expository paper. *A. G. Azpeitia.*

★ **Hoffman, A. J.; and Kruskal, J. B. Integral boundary points of convex polyhedra.** Linear inequalities and related systems, pp. 223–246. Annals of Mathematics Studies, no. 38. Princeton University Press, Princeton, N. J., 1956. $5.00.

Certain practical problems (e.g. transportation problems) lead to matrices with integral entries satisfying the following conditions: For any integral vector b, the extreme solutions of the inequalities (1) $Ax \geqq b$ ((2) $x \geqq 0$, $xA \geqq b$) have integral coordinates. The authors here are investigating properties of matrices satisfying (1) or (2) above. Theorem 1: A matrix of rank r satisfies (1) above if and only if in any set of r linearly independents rows of A the g.c.d. of the $r \times r$ determinants of these rows is 1. Theorem 2. A matrix satisfies (2) above if and only if all of its sub-determinants are 1, 0, or -1.

Matrices satisfying this last condition are said to have the unimodular property. The remainder of the paper is concerned with giving sufficient conditions for a matrix to have this property. The main result is the following: An oriented graph is called alternating if no two adjacent edges of a cycle are oriented in the same direction. Theorem 4: The incidence matrix of a set of directed paths of an alternating graph has the unimodular property. The result is applied to give other sufficient conditions generalizing those encountered in transportation and similar problems. *D. Gale* (Providence, R.I.).

Bellman, Richard. On some applications of dynamic programming to matrix theory. Illinois J. Math. **1** (1957), 297–301.

Iterative numerical methods are given for solving the linear equations $Ax=b$ and finding the largest eigenvalue of the matrix A, assumed to be positive definite symmetric and having all non-zero entries not more than k squares from the diagonal ($k=1, 2$). The methods given are more laborious than standard methods. *P. Wolfe.*

Osborn, Howard. The problem of continuous programs. Pacific J. Math. **6** (1956), 721–731.

Multi-stage decision processes may be either discrete or continuous, deterministic or stochastic. Corresponding to the classical problem of determining when the limit of solutions of a sequence of difference equations yields the solution of a differential equation, in the theory of dynamic programming [cf. Bellman, Bull. Amer. Math. Soc. **60** (1954), 503–515; MR **16**, 732] there is the problem of determining when a sequence of discrete dynamic programming processes converges to a continuous dynamic programming process.

The author establishes a general convergence theorem which is an extensive generalization of results previously obtained by the reviewer and others. The method used is closely related to the Cauchy-Lipschitz technique for establishing the existence and uniqueness of solutions of differential equations. The nonlinearity introduced by the decision process requires a more detailed treatment. *R. Bellman* (Santa Monica, Calif.).

See also: Arrow and Hurwicz, p. 954; Barankin, p. 954; Suppes, p. 954; Aitchison and Brown, p. 957; Davies, van Dunn and Hamaker, p. 958; Kalaba and Juncosa, p. 981.

Biology and Sociology

Trucco, Ernesto. Topological biology: A note on Rashevsky's transformation T. Bull. Math. Biophys. **19** (1957), 19–21.

Rashevsky, N. Outline of a possible mathematical approach to the problem of the effects of environment upon the incidence of some psychoneuroses. Bull. Math. Biophys. **19** (1957), 41–59.

See also: Crow and Kimura, p. 950; Dempster, p. 950; Neyman, Park and Scott, p. 950; Bartlett, p. 951; Bharucha-Reid, p. 951; Chiang, Hodges and Yerushalmy, p. 952; Kendall, p. 953; Taylor, p. 953; Anderson and Rubin, p. 954; Mosteller, p. 955; Solomon, p. 955.

Information and Communication Theory

Gel'fand, I. M.; and Yaglom, A. M. Computation of the amount of information about a stochastic function contained in another such function. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 3–52. (Russian)

The amount $J(\xi, \eta)$ of information about ξ, contained in η, where ξ and η are any two chance variables, is defined in this paper as follows: Let the ranges of ξ and η be divided, respectively, into $\Delta_1, \cdots, \Delta_n$, and $\Delta_1', \cdots, \Delta_m'$, a finite number of disjoint intervals, finite or not, open, closed, or half-open. Let $\xi(\Delta_1, \cdots, \Delta_n)$, $\eta(\Delta_1', \cdots, \Delta_m')$, be chance variables whose values are those of the serial number of the interval Δ_i ($i=1, \cdots, n$) and Δ_j' ($j=1, \cdots, m$) within ξ and η, respectively, lie. Let

$$J[\xi(\Delta_1, \cdots, \Delta_m), \eta(\Delta_1', \cdots, \Delta_n')]$$

be the usual Shannon definition of the amount of information in $\xi(\Delta_1, \cdots, \Delta_n)$ relative to $\eta(\Delta_1', \cdots, \Delta_m')$. Then $J(\xi, \eta)$ is defined as the supremum of

$$J[\xi(\Delta_1, \cdots, \Delta_n), \eta(\Delta_1', \cdots, \Delta_m')]$$

with respect to all possible systems of intervals $(\Delta_1, \cdots, \Delta_n)$ and $(\Delta_1', \cdots, \Delta_m')$. For two generalized stochastic processes [I. M. Gel'fand, Dokl. Akad. Nauk (N.S.) **100** (1955), 853–856; MR **16**, 938] the amount of information is defined analogously. The authors compute the amount of information about a Gaussian stochastic vector (resp. process), contained in another such vector (resp. process). Numerous examples are given. Necessary and sufficient conditions for J to be finite in various circumstances are given. A typical computational result is as follows: Let ξ and η be Gaussian with respective covariance matrices A and B and joint covariance matrix C. Then

$$J(\xi, \eta) = \tfrac{1}{2} \log \frac{\det A \cdot \det B}{\det C}$$

J. Wolfowitz (Ithaca, N.Y.).

Rozenblat-Rot, M. Theory of transmission of information through stochastic communication channels. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 202–205. (Russian)

The author gives a sequence of definitions and theo-

rems (without proof) which lead to a proof of Shannon's coding theorem for a noisy channel [Bell system Tech. J. 27 (1948), 379–423, 623–656; Th. 11; MR 10, 133]. The paper is already in summary form and does not lend itself to further summary. *J. Wolfowitz.*

Kalaba, R. E.; and Juncosa, M. L. Optimal design and utilization of communication networks. Management Sci. 3 (1956), 33–44.

The authors have tried to apply linear programming to the design and utilisation of communication networks and promise to give some numerical results in a later paper. *A. Jensen* (Lyngby).

See also: Fortet, p. 941.

Control Systems

Coales, J. F. An introduction to the study of non-linear control systems. J. Sci. Instrum. 34 (1957), 41–47.

An excellent expository account of the methods and underlying philosophy of nonlinear control system theory. *L. A. Zadeh* (New York, N.Y.).

★ **Айзерман, М.А. [Aĭzerman, M. A]. Лекции по теории автоматического регулирования. [Lectures in the theory of automatic regulation.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 428 pp. 12.10 rubles.

This book is based on a series of lectures which were given by the author to groups of scientists and engineers whose fields of specialization lay outside that of automatic control. Reflecting this fact, the exposition is on a fairly elementary level and makes no attempt to be comprehensive and up-to-date. Nevertheless, the book is so well written that a brief description of its contents seems desirable.

Like most other Soviet texts on automatic controls, the book begins with an extensive survey of various practical forms of servomechanisms. The characterization of automatic control systems by differential equations and the solution of these equations by the methods of the Fourier and Laplace transformations are treated next. This is followed by a thorough discussion of the stability of linear systems and the evaluation of their performance in terms of the least squares and other types of criteria. The book closes with a long chapter on nonlinear systems which is given over mainly to a discussion of the describing function technique. *L. A. Zadeh.*

Ioanin, Gh. Sur la synthèse des schémas aux conditions de travail données pour les éléments d'exécution. Com. Acad. R. P. Romîne 5 (1955), 935–942. (Romanian. Russian and French summaries)

Ulanov, G. M. Invariance up to ε in combined systems of automatic control. Dokl. Akad. Nauk SSSR (N.S.) 112 (1957), 253–256. (Russian)

Let $x(t)$ be the response of a feedback regulator to a disturbance $f(t)$, and let $x\mathrm{inv}(t)$ (inv=invariant) be the response of the regulator combined with a feed-forward corrective network. A system is said to be absolutely invariant if $x\mathrm{inv}(t)\equiv 0$; it is ε-invariant if $x\mathrm{inv}\,(t)=\varepsilon(t)x(t)$, where $\varepsilon(t)$ is a prescribed function of t. The author derives the parameters of the corrective network for the case where $\varepsilon(t)=\varepsilon t$, ε=positive constant. *L. A. Zadeh.*

Berezkin, E. N. Some questions of stability of motion. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. 11 (1956), no. 1, 23–31. (Russian)

The author studies the stability of control systems characterized by the equations

$$\dot{x}=y(1+f(x)),$$
$$\dot{y}=\varphi(x)+yF(x)+y^2\Phi(x)+y^3\Psi(x)+\cdots,$$

where $f(x)$, $\varphi(x)$, $F(x)$, $\Phi(x)$, $\Psi(x)$ are non-holomorphic continuous functions which vanish for $x=0$. By constructing suitable Lyapounov functions, the author obtains the following results: (1) If $\varphi(x)\equiv F(x)\equiv 0$, $\Phi(x)\not\equiv 0$ for $x\neq 0$, the trivial solution is unstable; (2) if $\varphi(x)\equiv 0$, $xF(x)>0$ or $xF(x)<0$ or $F(x)>0$ for $|x|<c$=const, $x\neq 0$, the trivial solution is unstable; (3) if $\varphi(x)\equiv 0$ and $F(x)<0$ for $x\neq 0$, $|x|<A$=sufficiently small constant, and

$$2F(x)-3y[xF(x)-2\Phi(x)/\textstyle\int_0^x \Phi(\xi)d\xi]<0$$

in some neighborhood of the origin, the trivial solution is stable; (4) if $\varphi(x)>0$ or $\varphi(x)<0$ for $x\varphi(x)>0$ for $x\neq 0$, $|x|<A$, the trivial solution is unstable; (5) if $x\varphi(x)<0$ for $x\neq 0$, $|x|<c$, and

$$2y^4\Psi(x)-y\varphi(y)F(y)-2[\varphi(x)-2y^3\Psi(x)]\int_0^x F(\tau)d\tau<0$$

in some neighborhood of the origin, the trivial solution is asymptotically stable; (6) if $F(x)\equiv 0$ and $\Psi(x)\leq 0$ for $x\neq 0$, the trivial solution is stable. The author's proofs of instability are based on Četaev's theorem [Dokl. Akad. Nauk SSSR (N.S.) 1 (1934), 529–531; Stability of motion Gostehizdat, Moscow, 1946]. *L. A. Zadeh.*

HISTORY, BIOGRAPHY

★ **Dijksterhuis, E. J. Archimedes.** Acta Hist. Sci. Nat. et Med., Edidit Bibliotheca Universitatis Hauniensis. Vol. 12. Ejnar Munksgaard, Copenhagen, 1956. The Humanities Press, Inc., New York, 1957. 422 pp. \$12.50.

This book will be welcome to mathematicians. As the author explains it takes a middle position between the works of Heath, who gives the arguments of Archimedes in modern notation, and the literal translation of Ver Eecke. The present author gives only the propositions in a literal translation, while the proofs are set forth in a symbolical notation specially devised for the purpose, which preserves the characteristic qualities of the classical argument. Moreover, the lemmas and the fundamental theorems in each treatise are separated in such a way that the general trend of the argument is always clear. The reviewer found these innovations very agreeable. Since two chapters, out of fifteen, deal with the life of Archimedes and with the history of his works and their manuscript tradition, the book will also be of interest to the humanist. *S. H. Gould* (Providence, R.I.).

Rosen, Edward. Maurolico was an abbot. Arch. Internat. Hist. Sci. 9 (1956), 349–350.

What appears on the titlepage of Maurolico's Opuscula mathematica (Venice, 1575) is "Abbatis", the genitive from of "Abbas", and not "Abbatus", as stated by H. Freudenthal (same Arch. VI (1953)), who had argued that such a title might refer to any member of the secular

clergy. Further evidence is given for the rank of Maurolico.

Mulcrone, T. F. The names of the curve of Agnesi. Amer. Math. Monthly **64** (1957), 359–361.

Fuller, A. W. Universal rectilinear dials. Math. Gaz. **41** (1957), 9–24.

An historical and mathematical discussion of several sundials, found in British and European Museums, of the type described by Regiomontanus (1436–1476).

★ **Newton, Isaac. Principiile matematice ale filozofiei naturale. [Mathematical principles of natural philosophy.]** Editura Academiei Republicii Populare Romîne, Bucureşti, 1956. 483 pp. Lei 30.65.

This is the first translation of Newton's Principia into Romanian. The book is large and attractively printed and the diagrams are good reproductions of the original ones.

★ **Dugas, René. A history of mechanics. Foreword by Louis de Broglie. Translated into English by J. R. Maddox.** Éditions du Griffon, Neuchâtel, Switzerland; Central Book Company, Inc., New York, 1957. 671 pp.

A translation of the book reviewed in MR **14**, 341.

Azpeitia, A. G. Isaac Barrow. Gac. Mat., Madrid (1) **9** (1956), 123–129. (Spanish)

A short biography with a photograph and with a discussion of the extent to which Barrow contributed to the fundamental principles of the calculus.

Bilimovič, A. D. A. M. Lyapunov in Odessa. Acad. Serbe Sci. Publ. Inst. Math. **9** (1956), 1–7. (Russian)

Pearson, E. S. John Wishart, 1898-1956. Biometrika **44** (1957), 1–8.

★ **Segre, Corrado. Opere.** A cura dell' Unione Matematica Italiana e col contributo del Consiglio Nazionale delle Ricerche. Volume I. Edizioni Cremonese, Roma, 1957. xii+445 pp. (1 plate). 4000 Lire.

This first volume contains 21 articles and a photograph of Segre. There is an eight-page preface by F. Severi. In the preface it is stated that ,,si decise, per ragioni di omogeneità ⋯ di non legarsi all' ordine cronologico ⋯. L'intensa operosità scientifica della sua prima giovinezza. ⋯ sulle ipersuperficie quadriche ⋯ e sulle applicazioni alla geometria della retta ⋯ non figurano nel presente volume. Con questo libro si è voluto invero aprire la serie dei volumi (saranno probabilmente quattro), ponendo anzitutto in evidenza ⋯ l'assurgere graduale di Segre alla visione della proprietà invarianti per trasformazione birazionali".

MISCELLANEOUS

Boas, R. P., Jr. "If this be treason ...". Amer. Math. Monthly **64** (1957), 247–249.

A plea for the traditional curriculum (or those parts which are most under attack) ending with "the traditional topics still serve a real purpose, even if it is not their ostensible purpose."

★ **McDowell, C. H. As hort dictionary of mathematics. Introduction by Henrietta O. Midonick. Part I, Arithmetic and algebra; Part II, Plane trigonometry and geometry.** Philosophical Library, New York, 1957. xiii+63 pp. $2.75.

An elementary dictionary.

Wopperer, E. Mathematik als Mathesis universalis. Math. Naturwiss. Unterricht **9** (1956/57), 437–447.

★ **Piene, Kay. School mathematics for universities and for life.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 318–324. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

★ **Daltry, C. T. Self-education by children in mathematics using Gestalt methods — i.e. learning-through-insight.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 297–304. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

★ **Kurepa, G. The role of mathematics and mathematician at present time.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 305–317. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

This is a preliminary report of the results of the "International Inquiry of the International Mathematical Instruction Commission (IMIC)", decided upon in Geneva, October 20, 1952.

MISSING PAGES ARE INDEX PAGES
WHICH HAVE BEEN PHOTOGRAPHED
AT THE BEGINNING OF THE VOLUME(S)